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Hilbert series and moduli spaces of k U(N) vortices

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Abstract: We study the moduli spaces of k U(N) vortices which are realized by the Higgs branch of a U(k) supersymmetric gauge theory. The theory has 4 supercharges and lives on k D1-branes in a N D3- and NS5-brane background. We realize the vortex moduli space as a $\mathbb{C}^*$ projection of the vortex master space. The Hilbert series is calculated in order to characterize the algebraic structure of the vortex master space and to identify the precise $\mathbb{C}^*$ projection. As a result, we are able to fully classify the moduli spaces up to 3 vortices.

Keywords: Supersymmetric gauge theory, D-branes, Differential and Algebraic Geometry, Superstring Vacua

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1 Introduction

The study of vortices has attracted much interest in the past since their first appearance in theoretical physics [1, 2]. More recently, the work in [3] presented a brane construction for a supersymmetric gauge theory with 4 supercharges describing k U(N) vortices. For simplicity, we will restrict to a $3d$ $\mathcal{N}=4$ theory in this work. The construction involves k D1-branes in a N D3-brane and NS5-brane background. Of our interest is the worldvolume theory of the D1-branes realized on the Higgs branch of the vortex theory. The D1-branes play the role of vortices and their worldvolume theory is effectively a $1d$ theory which is a dimensional reduction of a $2d$ $\mathcal{N}=(2,2)$ supersymmetric gauge theory. The vacuum moduli space of this theory is identified as the moduli space of k U(N) vortices on $\mathbb{C}$. The following work is interested in studying these moduli spaces using the construction in [3].

Vortex moduli spaces are complex projective spaces [3–13]. In this work, we express the vortex moduli spaces as partial $\mathbb{C}^*$ projections of vortex *master spaces*. These are spaces of mesonic and baryonic chiral operators which are invariant under the non-Abelian part of the gauge symmetry [14, 15].[1] For the case of k U(N) vortices, this is the SU(k) part of the U(k) gauge symmetry. Master spaces have been studied in various setups in string theory, with a particular focus on $4d$ $\mathcal{N}=1$ supersymmetric quiver gauge theories which can be represented by brane tilings [16–18]. The $\mathbb{C}^*$ projection of the vortex master space is along the remaining U(1) gauge symmetry and leads in general to a partially weighted projective space. This is precisely the full vortex moduli space we want to study in this work.

We take inspiration from the recent fruitful studies of moduli spaces of instantons on $\mathbb{C}^2$ [19–21]. The *ADHM construction* [22] for instanton moduli spaces arising from D3-D7 brane constructions resembles remarkably the construction of vortex moduli spaces. From this point of view, the vortex construction in [3] is often referred to as a $\frac{1}{2}$-ADHM construction for vortices. As it has been used for the study of instanton moduli spaces, we make use of *Hilbert series* [23–25] as a tool to analyze the algebraic structure of vortex moduli spaces. The use of Hilbert series combined with the use of vortex master spaces allows us to fully characterize the algebraic structure of vortex moduli spaces for up to 3 U(N) vortices on $\mathbb{C}$.

Hilbert series have been very successfully used to study vacuum moduli spaces of various supersymmetric gauge theories. They are partition functions of gauge invariant chiral operators. Hilbert series have been used for instance to shed light on moduli spaces of SQCD with classical gauge groups [26] and toric moduli spaces of brane tilings [27–29].

We first compute the Hilbert series of the vortex master space by taking gauge invariance under the SU(k) non-Abelian part of the gauge symmetry. The Hilbert series allows us to identify the full algebraic structure of the vortex master space, including information about its generators and quadratic relations satisfied by the generators. In general, we observe and verify that the k U(N) vortex master space is a non-compact singular Calabi-Yau cone of complex dimension $kN+1$. The algebraic variety of the vortex master space is

[1] Strictly speaking, the master space is the space of invariants under the non-Abelian part of the gauge symmetry *and* under the F-term constraints which are obtained by the superpotential of the theory. We will later see that the vortex theory does not have relevant a superpotential.

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weighted under the remaining U(1) symmetry. These weights are part of the $\mathbb{C}^*$ projection which lifts the vortex master space to the full now partially compact vortex moduli space of complex dimension kN. This work for the first time uses Hilbert series to fully classify the vortex moduli spaces up to 3 U(N) vortices and presents the Hilbert series for 4 U(1) and U(2) vortices.

The outline of the paper is as follows. Section 2 reviews the analysis of instanton moduli spaces from the ADHM construction and summarizes the similarities to the vortex construction in [3]. The section introduces the computation for the Hilbert series of the vortex master space and explains its $\mathbb{C}^*$ projection into the full vortex moduli space. Using the techniques presented in section 2, sections 3 to 5 present the a classification of vortex moduli spaces up to 3 vortices for any U(N). Section 6 illustrates with Hilbert series for 4 U(1) and U(2) vortices that the classification scheme we introduce here with this paper is generalizeable to any number of U(N) vortices. Finally, section 7 summarizes the unrefined Hilbert series for our classification of vortices as well as uses a new compact form of presenting character expansions of Hilbert series.

2 Background

In this section, we review the theoretical background on vortices based on [3]. An outline is given for the brane construction of vortices and the corresponding quiver diagram of the worldvolume theory. We are interested in the Higgs branch moduli space of the vortex theory which we will describe as a $\mathbb{C}^*$ projection of the master space. The Hilbert series is obtained in order to characterize the master space, the $\mathbb{C}^*$ projection and ultimately the vortex moduli space itself.

The quiver theory for k U(N) vortices is strikingly similar to the ADHM construction of k U(N) instantons. Given that the Higgs branch moduli space of k U(N) instantons has been extensively studied with the help of Hilbert series [23–25], let us review the study of instanton moduli spaces as a warm-up for vortices.

2.1 Vortices from instantons

k U(N) instantons revisited. The moduli space of instantons on $\mathbb{C}^2$ is the Higgs branch of a $\mathcal{N}=2$ supersymmetric gauge theory in $3+1$ dimensions. It is the worldvolume theory of k D3-branes in a N D7-brane background. At the Higgs branch of the theory of k U(N) instantons, the k D3-branes are on top of the N D7-branes, the position of the D3-branes in the D7-branes being in $\mathbb{C}^2$.

The worldvolume theory is a $4d$ $\mathcal{N}=2$ quiver gauge theory. Its quiver diagram consists of a $\mathcal{N}=2$ hypermultiplet and an $\mathcal{N}=2$ adjoint hypermultiplet. There is a U(k) vector multiplet with a U(N) global symmetry. The U(1) of U(N) can be absorbed into the local U(k) giving us a two noded quiver where one node corresponds to the local U(k) and the other to the global SU(N).

The quiver and superpotential can be expressed in terms of $\mathcal{N}=1$ language by decomposing the $\mathcal{N}=2$ hyper and vector multiplets into $\mathcal{N}=1$ chiral and vector multiplets. The resulting quiver diagram is shown in figure 1. The fields carry the gauge and global charges

	$\mathrm{U}(k)_{\mathrm{gauge}}$		$\mathrm{U}(N)_{\mathrm{global}}$		
	$\mathrm{SU}(k)_w$	$\mathrm{U}(1)_z$	$\mathrm{SU}(N)_x$	$\mathrm{SU}(2)_{\mathbb{C}^2}$	$\mathrm{U}(1)_r$
Φ	$[1,0,\ldots,0,1]_w+1$	0	$[0,\ldots,0]_x$	$[0]_q$	0
ϕ_1,ϕ_2	$[1,0,\ldots,0,1]_w+1$	0	$[0,\ldots,0]_x$	$[1]_q$	1
Q	$[1,0,\ldots,0]_w$	$+1$	$[0,\ldots,0,1]_x$	$[0]_q$	1
$\widetilde{Q}$	$[0,\ldots,0,1]_w$	-1	$[1,0,\ldots,0]_x$	$[0]_q$	1

Table 1. Charges carried by the fields of the k $\mathrm{U}(N)$ instanton quiver. The $\mathrm{U}(1)$ of the global $\mathrm{U}(N)$ has been absorbed into the local $\mathrm{U}(k)$ for simplicity. The fugacity for the $\mathrm{SU}(2)_R$ charge is t and for the $\mathrm{SU}(2)_{\mathbb{C}^2}$-charge is q.

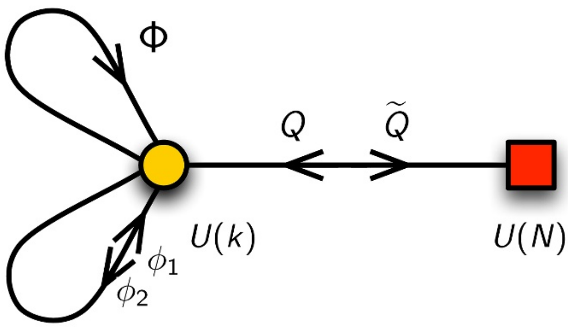

Figure 1. $\mathcal{N}=1$ quiver for the theory describing k $\mathrm{U}(N)$ instantons in $\mathbb{C}^2$.

as shown in table 1.[2] Spacetime is broken to $\mathbb{R}^{3,1}\times\mathbb{C}^2\times\mathbb{C}$ where $\mathbb{C}^2$ has an isometry of $\mathrm{U}(2)=\mathrm{SU}(2)_{\mathbb{C}^2}\times\mathrm{U}(1)_r$ where the $\mathrm{U}(1)_r$ is the Cartan element of $\mathrm{SU}(2)_R$. From the quiver in figure 1, the superpotential can be written as follows,

$$W=\mathrm{Tr}(Q\cdot\Phi\cdot\widetilde{Q}+\phi_1\cdot\Phi\cdot\phi_2-\phi_2\cdot\Phi\cdot\phi_1)\,. \tag{2.1}$$

For the Higgs branch, the k D3-branes are on top of the N D7-branes and hence we have $\langle\Phi\rangle=0$.

The ADHM data [22] is summarized by the quiver fields and the F-terms originating from the superpotential. Following the construction, we analyze the Higgs branch of the above supersymmetric quiver gauge theory as the moduli space of k $\mathrm{U}(N)$ instantons. The **Hilbert series** [23–25] is a generating function of gauge invariant operators and can be used to characterise moduli spaces such as the instanton moduli space which we are studying here. The Hilbert series encodes information about the generators and relations formed among the generators. This information can be extracted from the Hilbert series

[2] *Notation:* we use for the character of irreducible representations of $\mathrm{SU}(N)$ the notation $[a_1,\ldots,a_{N-1}]_x$, where the set of positive integers $\{a_i\}$ with $1\le i\le N-1$ are the highest weight of the representation. The corresponding Young diagram made of rows of length λ_i can be found using the identification $a_i=\lambda_i-\lambda_{i+1}$. The subscript x in $[a_1,\ldots,a_{N-1}]_x$ indicates which variable is used for the set of fugacities that count the weights in the character. For example, the fundamental of SU(3) is written as $[1,0]_x=x_1+\frac{x_2}{x_1}+\frac{1}{x_2}$.

through various techniques which we are outlining here with the example of the instanton moduli space.

The generating function which counts all possible products of the quiver fields in table 1 is given by the following plethystic exponential[3] as first outlined in [19],

$$f_1 = \mathrm{PE}\Big[([1,0,\ldots,0,1]_w+1)[1]_q t \\ + [1,0,\ldots,0]_w z[0,\ldots,0,1]_x t + [0,\ldots,0,1]_w z^{-1}[1,0,\ldots,0]_x t\Big] \,. \tag{2.2}$$

The above generating function does not take into account redundancies under F-term relations. The F-term takes the form

$$\mathcal{F} := \partial_\Phi W = Q\cdot \widetilde{Q} + \phi_1\cdot\phi_2 - \phi_2\cdot\phi_1 \,. \tag{2.3}$$

The F-term carries the $\mathrm{U}(1)_r$ charge of the superpotential and also transforms under the adjoint representation of the gauge group. As such, the following generating function counting contributions from from the F-terms should be added from from the generating function in (2.2),

$$f_2 = \mathrm{PE}\Big[-([1,0,\ldots,0,1]_w+1)t^2\Big] \,. \tag{2.4}$$

Overall, $f_1 f_2$ is a generating function counting all gauge invariant and non-gauge invariant operators from quiver fields in table 1 subject to F-term constraints. The Hilbert series of the k $\mathrm{U}(N)$ instanton moduli space $\mathcal{M}^{inst}_{k,N}$ can be calculated when one integrates out the gauge charge dependence of the above plethystic exponentials,

$$g(x,q,t;\mathcal{M}^{inst}_{k,N}) = \oint \mathrm{d}\mu_{\mathrm{SU}(N)}\mathrm{d}\mu_{\mathrm{U}(1)}\ f_1 f_2 \,, \tag{2.5}$$

where $\mathrm{d}\mu_{\mathrm{SU}(N)}$ and $\mathrm{d}\mu_{\mathrm{U}(1)}$ are the Haar measures of $\mathrm{SU}(N)$ and $\mathrm{U}(1)$ respectively.

From the instanton to the vortex. It has been outlined in [3] and further evaluated in [6, 7] and consecutive papers that the instanton theory is related to the theory of k $\mathrm{U}(N)$ vortices. The worldvolume theory of the vortices is effectively a $1d$ theory which is a dimensional reduction of a $2d$ $\mathcal{N}=2$ superymmetric gauge theory. The construction of the vortex moduli space from this quiver theory resembles the ADHM construction of the instanton moduli space. In literature, the construction for vortices is also referred to as a $\frac{1}{2}$-ADHM construction.

Considering the quiver for the ADHM construction of instantons, the construction for the vortex moduli space precisely requires half of the quiver field content: a single fundamental Q between $\mathrm{U}(k)$ and $\mathrm{U}(N)$, and a single adjoint $\phi\equiv\phi_1$. The combination of $\{\phi, Q\}$ precisely forms the field content of the vortex theory. Let us consider first the brane construction for the k $\mathrm{U}(N)$ vortex in the following section.

[3]For a multivariate function $f(t_1,\ldots,t_n)$, the *plethystic exponential* is defined as $\mathrm{PE}\left[f(t_1,\ldots,t_n)\right] = \exp\left[\sum_{k=1}^{\infty}\frac{f(t_1^k,\ldots,t_n^k)}{k}\right]$. The PE acts as generator for symmetrisation of $f(t_1,\ldots,t_n)$. We refer to [30] for a mathematical exposition of this property.

	0	1	2	3	4	5	6	7	8	9
NS5	×	×	×	×	×	×				
N D3	×	×	×				×			
k D1										×

Table 2. k D1-branes and N D3-branes which are suspended between NS5-branes. The theory for this brane construction has 4 supercharges. For simplicity we restrict to a $3d$ $\mathcal{N}=4$ U(N) theory in this work.

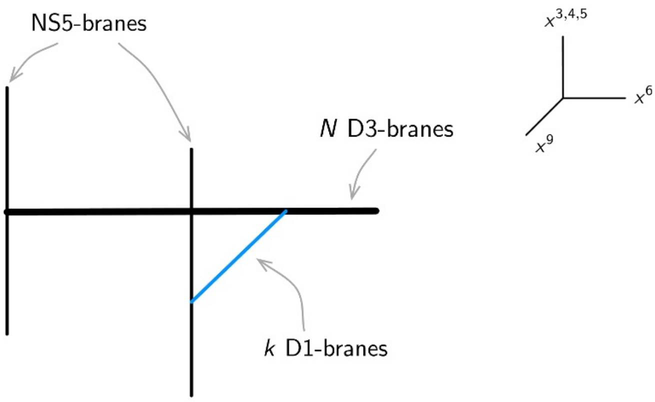

Figure 2. Brane construction for k U(N) vortices. It represents the Higgs branch of a $3d$ $\mathcal{N}=4$ U(N) Yang-Mills Higgs theory.

2.2 The brane construction and the vortex moduli space

Vortices in Type IIB string theory are realized by a construction of k D1-branes and N D3-branes which are suspended between NS5-branes [3, 31]. The theory for this brane construction is a theory with 4 supercharges. For simplicity, we take it here to be a $3d$ $\mathcal{N}=4$ U(N) Yang-Mills Higgs theory. Table 2 shows the brane picture in 9+1 dimensions.

The BPS vortices are represented by the k D1-branes. We are interested in the Higgs branch of the Yang-Mills Higgs theory where the FI parameter ζ is non-zero. The mass and size of the vortices scale respectively as $M_v \sim \zeta$ and $M_\gamma \sim \zeta^{-1/2}$. The finite FI parameter ζ relates to a decoupling of a NS5 brane from the rest of the construction. Between the decoupled NS5-brane and N D3-branes are the k suspended D1-branes as shown in figure 2.

In order to describe the vortex moduli space, we consider the worldvolume theory of the k D1-branes. The worlvolume theory is effectively a $1d$ theory which is a dimensional reduction of a $2d$ $\mathcal{N}=(2,2)$ supersymmetric gauge theory as mentioned in section 2.1. The field content $\{\phi, Q\}$ of the theory consists of a complex scalar adjoint ϕ which parameterizes the position of the k D1-branes in the x_1-x_2 plane. It also contains a fundamental Q in the U(k) gauge group arising from D1-D3 strings. The vector multiplet contains the gauge field and complex scalar fields parameterizing the degrees of freedom of the D1-branes in the $x^{3,4,5}$ directions.

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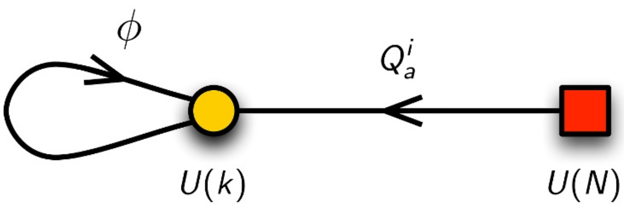

Figure 3. Quiver diagram of the k U(N) vortex theory.

The Higgs branch $\mathcal{M}^{vort}_{k,N} \equiv \mathcal{V}_{k,N}$ of the vortex theory is determined by a Kähler quotient of $\mathbb{C}^{kN+k^2}$ parameterized by the $k \times k$ matrices ϕ and $k \times N$ matrices Q. There is no superpotential and therefore no F-terms. The D-terms are given by

$$D^m{}_n = \phi^m{}_i (\phi^\dagger)^i{}_n - [Q, Q^\dagger]^m{}_n - r\delta^m{}_n\,, \tag{2.6}$$

where r is the finite non-zero FI parameter of the vortex theory for the Higgs branch $\mathcal{V}_{k,N}$. The D-terms impose k^2 constraints and with the U(k) gauge group which results in a further k^2 reduction, the total real dimension of the Higgs branch reduces to

$$\dim_{\mathbb{R}} (\mathcal{V}_{k,N}) = 2(k^2 + kN) - k^2 - k^2 = 2kN\,. \tag{2.7}$$

The complex dimension is $\dim_{\mathbb{C}} = kN$, and is precisely half the dimension of the k U(N) instanton moduli space which we discussed in section 2.1.

We demand the D-term equations to be satisfied. For the vortex master space, gauge invariance is only taken for the non-Abelian SU(k) part of the U(k) symmetry. The remaining U(1) has a FI-term in the D-term equations, and as such for non-zero FI-terms the D-term equations are set to be constant. This together with U(1) gauge invariance amounts to complex rescaling of the SU(k) invariant baryonic operators along the U(1) direction. This is the partial $\mathbb{C}^*$ projection of the vortex master space in order to obtain the full moduli space of the vortices. In the following section, we review the $\frac{1}{2}$-ADHM construction similar to the construction of the instanton moduli space, and outline the method of using Hilbert series to study partial projective spaces as moduli spaces of k U(N) vortices.

2.3 The quiver and the Hilbert series

Vortex quiver. The field content of the vortex theory consisting of the adjoint scalar ϕ and the fundamental Q in the gauge group U(k) can be represented by a 2-noded quiver diagram as shown in figure 3. The two nodes of the quiver diagram represent the gauge group U(k) and the global symmetry group U(N). The transformation laws of the fields are summarized in table 3.

There is no superpotential and hence no corresponding F-term equations. For gauge invariance, we demand invariance under the non-Abelian SU(k) part of the U(k) gauge symmetry. The remaining U(1) symmetry has a FI-term in the D-term equations. They are set to be a constant due to the FI-term which combined with U(1) gauge invariance amounts

	$\mathrm{U}(k)_{\mathrm{gauge}}$		$\mathrm{U}(N)_{\mathrm{global}}$			
	$\mathrm{SU}(k)_w$	$\mathrm{U}(1)_z$	$\mathrm{SU}(N)_x$	$\mathrm{U}(1)_q$	$\mathrm{U}(1)_r$	$\mathrm{U}(1)_s$
ϕ	$[1,0,\ldots,0,1]_w+1$	0	$[0,\ldots,0]_x$	0	1	1
Q^i_a	$[0,\ldots,0,1]_w$	$+1$	$[1,0,\ldots,0]_x$	-1	1	0

Table 3. Quiver fields of the vortex theory and their transformation properties.

to complex rescaling of the $\mathrm{SU}(k)$ invariant baryonic operators along the $\mathrm{U}(1)$ direction. From the quiver in figure 3 we observe that only the fundamental Q transforms under the $\mathrm{U}(1)$ parts of both the gauge $\mathrm{U}(k)$ and global $\mathrm{U}(N)$ symmetries. These transformations under the two $\mathrm{U}(1)$'s are not independent and as such we can absorb the $\mathrm{U}(1)$ of the local symmetry $\mathrm{U}(k)$ into the $\mathrm{U}(1)$ of the global $\mathrm{U}(N)$ without any loss of generality.

Vortex moduli space. The vortex moduli space $\mathcal{V}_{k,N}$ for k $\mathrm{U}(N)$ vortices is a partially weighted projective space originating from a partial $\mathbb{C}^*$ projection of the vortex master space $\mathcal{F}^\flat_{k,N}$. We denote this relationship as follows,

$$\mathcal{V}_{k,N} \equiv \mathbb{WP}_{\mathrm{U}(1)}[\mathcal{F}^\flat_{k,N}]\,. \tag{2.8}$$

The projection of the master space $\mathcal{F}^\flat_{k,N}$ is along the $\mathrm{U}(1)$ part of the $\mathrm{U}(k)$ gauge symmetry.

The vortex master space $\mathcal{F}^\flat_{k,N}$ is a space of gauge invariant operators which are invariant under the $\mathrm{SU}(k)$ part of the gauge symmetry. We make use of the Hilbert series to identify the algebraic structure of the master space. By identifying the $\mathrm{U}(1)$ gauge charges carried by the generators of $\mathcal{F}^\flat_{k,N}$, we can specify the projection into the full vortex moduli space $\mathcal{V}_{k,N}$ as a partially weighted projected space.

We make use of the following notation to describe the projection of $\mathcal{F}^\flat_{k,N}$ along the $\mathrm{U}(1)$ gauge charges,

$$\mathcal{V}_{k,N} = \mathcal{F}^\flat_{k,N}/\{\alpha_1 \simeq \lambda^{w_1}\alpha_1\,, \ldots\,, \alpha_n \simeq \lambda^{w_n}\alpha_n\}\,, \tag{2.9}$$

where α_i are the generators of $\mathcal{F}^\flat_{k,N}$, w_i are the $\mathrm{U}(1)$ gauge charges, and λ is the $\mathbb{C}^*$ parameter. The master space is determined by the following symplectic quotient

$$\mathcal{F}^\flat_{k,N} = \mathbb{C}^c//\mathrm{SU}(k)\,, \tag{2.10}$$

where c is the dimension of the freely generated space of all quiver fields. Note that there is no superpotential, and hence the master space here is determined without an ideal made of F-term constraints.

The dimension of the master space $\mathcal{F}^\flat_{k,N}$ for the k $\mathrm{U}(N)$ vortex theory is

$$\dim_{\mathbb{C}} \mathcal{F}^\flat_{k,N} = kN + 1\,, \tag{2.11}$$

which reduces via the $\mathbb{C}^*$ projection to the dimension of the vortex moduli space

$$\dim_{\mathbb{C}} \mathcal{V}_{k,N} = kN\,. \tag{2.12}$$

Figure 4 shows a graphical description of the vortex moduli space.

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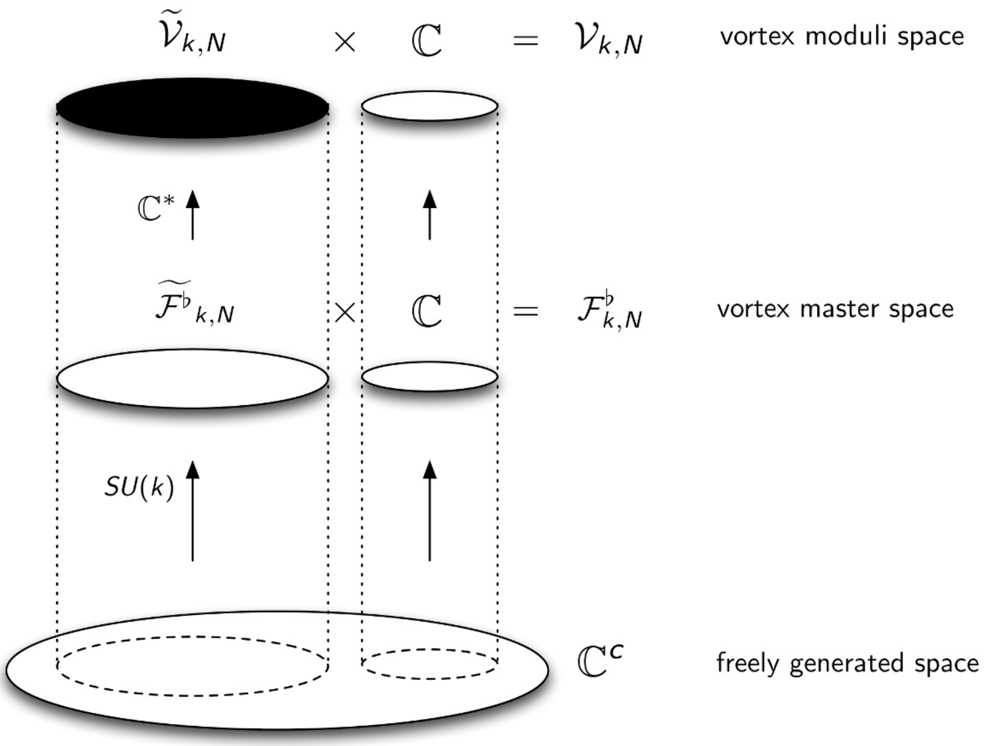

Figure 4. *Vortex moduli space.* The freely generated space of quiver fields is lifted to the vortex master space $\mathcal{F}^{\flat}_{k,N}$ by quotienting out the $\mathrm{SU}(k)$ gauge charges. The full vortex master space $\mathcal{F}^{\flat}_{k,N}$ contains the irreducible part of the master space $\widetilde{\mathcal{F}^{\flat}}_{k,N}$ and the center of mass position factor $\mathbb{C}$. The $\mathbb{C}^*$ projection leads to the vortex moduli space $\mathcal{V}_{k,N}$.

Vortex Hilbert series. For the Hilbert series computation, we have the following plethystic exponentials that contribute to the Hilbert series. For the fundamental Q^i_a we have the contribution

$$\mathrm{PE}\Big[[1,0,\dots,0]_x[0,\dots,0,1]_w z\ t\Big] = \frac{1}{\prod_{i=1}^{k-1}\prod_{j=1}^{N-1}(1-\frac{x_i}{x_{i-1}}\frac{w_{j-1}}{w_j}\ z\ t)}, \tag{2.13}$$

where $x_0 = x_k = 1$ and $w_0 = w_N = 1$. The contribution from the adjoint ϕ is the plethystic exponential

$$\mathrm{PE}\Big[([1,0,\dots,0,1]_w+1)s\Big] = \frac{1}{1-s}\mathrm{PE}\Big[[1,0,\dots,0,1]_w\ s\Big]. \tag{2.14}$$

The Molien integral for the Hilbert series of the master space of k $\mathrm{U}(N)$ vortices is

$$\begin{aligned} g_{k,\mathrm{U}(N)}(s,t,x;\mathcal{F}^{\flat}_{k,N}) &= \oint \mathrm{d}\mu_{\mathrm{SU}(k)}\mathrm{PE}\Big[[1,0,\dots,0]_x[0,\dots,0,1]_w\ z\ t \\ &\qquad\qquad +([1,0,\dots,0,1]_w+1)\ s\Big] \\ &= \frac{1}{1-s}\oint \mathrm{d}\mu_{\mathrm{SU}(k)}\mathrm{PE}\Big[[1,0,\dots,0]_x[0,\dots,0,1]_w\ z\ t \\ &\qquad\qquad +[1,0,\dots,0,1]_w\ s\Big], \end{aligned} \tag{2.15}$$

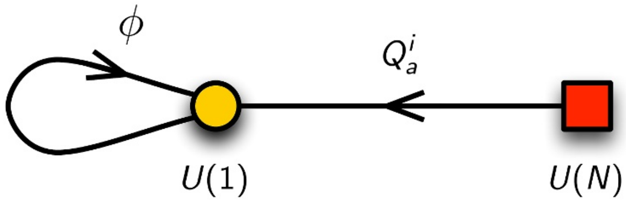

Figure 5. Quiver diagram of the 1 U(N) vortex theory.

where $\mathrm{d}\mu_{\mathrm{SU}(k)}$ is the Haar measure for SU(k). Note that the $\frac{1}{1-s}$ prefactor corresponds to a $\mathbb{C}$ factor of the vortex moduli space which parameterizes the centre of mass of the vortices.

3 1 U(N) vortex on $\mathbb{C}$

As a pedagogical introduction, we go through the example of 1 U(N) vortices and re-derive known results from [3] by making use of Hilbert series and the notion of vortex master spaces. The quiver diagram of the 1 U(N) vortex theory is shown in figure 5.

The moduli space. The Hilbert series of the 1 U(N) vortex master space $\mathcal{F}^\flat_{1,N}$ does not require a Molien integral. The U(1) gauge charges are kept in the Hilbert series in order to identify the $\mathbb{C}^*$ projection weights for the generators of the master space. Recall that the generators of the master space $x_1, \ldots, x_d$ play the role of the projection coordinates, where the $\mathbb{C}^*$ projection is given by

$$(x_1, \ldots, x_d) \simeq (\lambda^{w_1} x_1, \ldots, \lambda^{w_d} x_d)\,. \tag{3.1}$$

λ is the $\mathbb{C}^*$ parameter and $w_1, \ldots, w_d$ are the projection weights along the corresponding projection coordinates $x_1, \ldots, x_d$. These U(1) weights $w_1, \ldots, w_d$ are used to partially project the master space $\mathcal{F}^\flat_{1,N}$ into the vortex moduli space $\mathcal{V}_{1,N}$. The vortex moduli space then takes the following form in analogy to the above projection,

$$\mathcal{V}_{1,N} = \mathcal{F}^\flat_{1,N} / \{x_1 \simeq \lambda^{w_1} x_1, \ldots, x_d \simeq \lambda^{w_d} x_d\}\,, \tag{3.2}$$

where $\mathcal{F}^\flat_{1,N}$ is parameterized by its generators $x_1, \ldots, x_d$.

The Molien integral and Hilbert series. The U(1) gauge charge is carried by the fugacity t corresponding to Q. Accordingly, in terms of the U(1) gauge and SU(N) global symmetries of the vortex theory, the Hilbert series for the 1 U(N) vortex master space can be written as,

$$g(t, s, x; \mathcal{F}^\flat_{1,N}) = \mathrm{PE}\Big[[1, 0, \ldots, 0]_x t + s\Big] = \frac{1}{1-s} \mathrm{PE}\Big[[1, 0, \ldots, 0]_x t\Big]\,, \tag{3.3}$$

where $[1, 0, \ldots, 0]_x$ is the fundamental representation of SU(N).

Center of mass position. Note that the overall factor $\frac{1}{1-s}$ in (3.3) corresponds to the centre of mass position of the vortex in $\mathbb{C}$. We can ignore the overall position of the vortex and obtain the Hilbert series of the *reduced* master space $\widetilde{\mathcal{F}^\flat}_{1,N}$ excluding the centre of mass position. The Hilbert series of the reduced master space is given by

$$g(t, s, x; \widetilde{\mathcal{F}^\flat}_{1,N}) = (1-s) \times g(t, s, x; \mathcal{F}^\flat_{1,N}) \,. \tag{3.4}$$

Note that the centre of mass is not involved in the projection of the master space into the vortex moduli space because it is not charged under the U(1) gauge symmetry. Accordingly, we will later make use of this fact and project the reduced master space $\widetilde{\mathcal{F}^\flat}_{1,N}$ into the reduced vortex moduli space $\mathcal{V}_{1,N} = \widetilde{\mathcal{V}}_{1,N} \times \mathbb{C}$. In the following discussion, we, for simplicity, refer interchangeably to $\widetilde{\mathcal{V}}_{1,N}$ and $\mathcal{V}_{1,N}$ as the moduli space of the 1 U(N) vortex on $\mathbb{C}$.

Plethystic logarithm. The plethystic logarithm of the Hilbert series extracts information about the generators of the moduli space and the relations formed by them. It is defined for a multivariate function $f(t_1, \ldots, t_n)$ as

$$\mathrm{PL}\left[f(t_1, \ldots, t_n)\right] = \sum_{k=1}^{\infty} \frac{\mu(k)}{k} \log\left[f(t_1^k, \ldots, t_n^k)\right] \,, \tag{3.5}$$

where $\mu(k)$ is the Möbius function. The plethystic logarithm is the inverse function of the plethystic exponential. If the expansion of the plethystic logarithm of a Hilbert series is finite with just positive terms, the corresponding moduli space is a *freely generated space*. If the finite expansion contains both positive and negative terms, the moduli space is a *complete intersection* generated by a finite number of generators subject to a finite number of relations. If the expansion is infinite, the moduli space is a non-complete intersection. The first positive terms of the expansion refer to generators of the moduli space. All higher order terms refer to relations among generators and relations among relations called *syzygies*. For a comprehensive review, the reader is referred to [14] and references therein. In this work, we concentrate on the application of the plethystic logarithm in order to study the structure of vortex moduli spaces.

3.1 1 U(N) vortex on $\mathbb{C}$

The first few examples of the refined Hilbert series for the master space of the 1 U(N) vortex theory are as follows,

$$\begin{aligned} g(t, s, x; \widetilde{\mathcal{F}^\flat}_{1,1}) &= \frac{1}{1-t} \,, \\ g(t, s, x; \widetilde{\mathcal{F}^\flat}_{1,2}) &= \frac{1}{(1-xt)(1-\frac{1}{x}t)} \,, \\ g(t, s, x; \widetilde{\mathcal{F}^\flat}_{1,3}) &= \frac{1}{(1-x_1 t)(1-\frac{x_2}{x_1}t)(1-\frac{1}{x_2}t)} \,, \end{aligned} \tag{3.6}$$

The plethystic logarithms of the above Hilbert series are

$$\begin{aligned}
\mathrm{PL}\left[g(t,s,x;\widetilde{\mathcal{F}^\flat}_{1,1})\right] &= t\,,\\
\mathrm{PL}\left[g(t,s,x;\widetilde{\mathcal{F}^\flat}_{1,2})\right] &= xt+\frac{1}{x}t\,,\\
\mathrm{PL}\left[g(t,s,x;\widetilde{\mathcal{F}^\flat}_{1,2})\right] &= x_1t+\frac{x_2}{x_1}t+\frac{1}{x_2}t\,.
\end{aligned} \tag{3.7}$$

We observe that the master spaces $\mathcal{F}^\flat_{1,N}$ are freely generated spaces of dimension $N+1$,

$$\mathcal{F}^\flat_{1,N}=\mathbb{C}^{N+1}\,. \tag{3.8}$$

As a character expansion in terms of characters of irreducible representations of the global $\mathrm{SU}(N)$ symmetry, the Hilbert series is

$$g(t,s,x;\widetilde{\mathcal{F}^\flat}_{1,N}) = \sum_{n=0}^{\infty}[n,0,\dots,0]_x t^n\,, \tag{3.9}$$

where the PE in (3.3) acts as a function generating symmetric products of the fundamental representation of $\mathrm{SU}(N)$.

The plethystic logarithm is the inverse function of the plethystic exponential. We recall that the full master space $\mathcal{F}^\flat_{1,N}=\mathbb{C}\times\widetilde{\mathcal{F}^\flat}_{1,N}$ is generated via ϕ counted by s and Q counted by t in the vortex quiver diagram and the corresponding Hilbert series is

$$g(t,s,x;\mathcal{F}^\flat_{1,N}) = \mathrm{PE}\left[[1,0,\dots,0]_x t+s\right] = \frac{1}{1-s}\mathrm{PE}\left[[1,0,\dots,0]_x t\right]\,. \tag{3.10}$$

Since the plethystic logarithm is the inverse function of the plethystic exponential, we have

$$\mathrm{PL}\left[g(t,s,x;\mathcal{F}^\flat_{1,N})\right] = \mathrm{PL}\left[g(t,s,x;\widetilde{\mathcal{F}^\flat}_{1,N})\right]+s = [1,0,\dots,0]_x t+s\,. \tag{3.11}$$

In general, this analogy between the plethystic exponential and logarithm can be utilised to identify the generators for more complicated moduli spaces.[4] For the simple case of $\widetilde{\mathcal{F}^\flat}_{1,N}$ here, the generators can be summarised as follows,

$$[1,0,\dots,0]_x t \to Q^i\,. \tag{3.12}$$

We expect that the 1 $\mathrm{U}(N)$ vortex moduli space $\mathcal{V}_{1,N}$ has complex dimension N. The $\mathbb{C}^*$ projection reduces the dimension of the master space $\mathcal{F}^\flat_{1,N}$ by 1. According to the Hilbert series in (3.11), the Q^i are the only objects carrying a $\mathrm{U}(1)$ charge, which is interpreted as $w_i=1$ for all Q^i. Therefore, the $\mathbb{C}^*$ projection of the reduced master space $\widetilde{\mathcal{F}^\flat}_{1,N}$ is given by

$$\widetilde{\mathcal{V}}_{1,N} = \widetilde{\mathcal{F}^\flat}_{1,N}/\{Q^i\simeq\lambda Q^i\}\,, \tag{3.13}$$

which implies that the 1 $\mathrm{U}(N)$ vortex moduli space is

$$\widetilde{\mathcal{V}}_{1,N}=\mathbb{CP}^{N-1}\,, \tag{3.14}$$

in other words, as the complex projective space of dimension $N-1$. This result is known for example from [3].

[4]For a complete review of techniques used here, the reader is referred to [14] and references therein.

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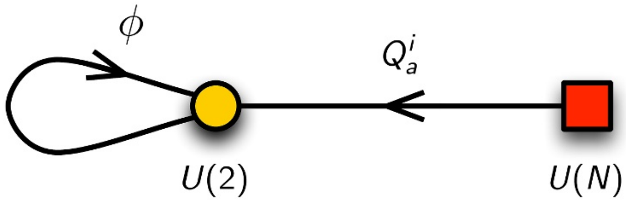

Figure 6. Quiver diagram of the 2 U(N) vortex theory.

4 2 U(N) vortices on $\mathbb{C}$

We proceed here to the case of 2 U(N) vortices on $\mathbb{C}$ by generalizing the previous analysis on 1 U(N) vortices with Hilbert series and vortex master spaces. The quiver diagram of the 2 U(N) vortex theory is shown in figure 6.

The moduli space. We recall that the 2 U(N) vortex moduli space $\mathcal{V}_{2,N}$ is a partial $\mathbb{C}^*$ projection of the corresponding master space $\mathcal{F}^\flat_{2,N}$. The generators $x_1,\ldots,x_d$ of $\mathcal{F}^\flat_{2,N}$ are projection coordinates with corresponding projection weights $w_1,\ldots,w_d$. Under the $\mathbb{C}^*$ projection, one has

$$(x_1,\ldots,x_d) \simeq (\lambda^{w_1}x_1,\ldots,\lambda^{w_d}x_d)\,, \tag{4.1}$$

where λ is the $\mathbb{C}^*$ parameter. Analogously, the master space is $\mathbb{C}^*$ projected as follows to give the vortex moduli space

$$\mathcal{V}_{2,N} = \mathcal{F}^\flat_{2,N}/\{x_1 \simeq \lambda^{w_1}x_1,\ldots,x_d \simeq \lambda^{w_d}x_d\}\,, \tag{4.2}$$

where the generators $x_1,\ldots,x_d$ parameterize the master space $\mathcal{F}^\flat_{2.N}$.

The Molien integral and Hilbert series. The computation for the Hilbert series of the 2 U(N) vortex master space $\mathcal{F}^\flat_{2,N}$ requires an integral over the SU(2) gauge charges of the vortex theory. The Molien integral is

$$g(t,s,x;\mathcal{V}_{2,N}) = \oint \mathrm{d}\mu_{\mathrm{SU}(2)}\ \mathrm{PE}\Big[[1,0,\ldots,0]_x[1]_w t + [2]_w s\Big]\,, \tag{4.3}$$

where $\mathrm{d}\mu_{\mathrm{SU}(2)}$ is the Haar measure of SU(2). The entries in the plethystic exponential correspond to Q^i_α transforming as $[1,0\ldots,0]_x[1]_w t$, and ϕ transforming in $[2]_w s$.

Center of mass position. Recall that the character for the adjoint representation of SU(2) is $[2]_w = w^2+1+w^{-2}$. Accordingly, the integrand in (4.3) can be rewritten as

$$\mathrm{PE}\Big[[1,0,\ldots,0]_x[1]_w t + [2]_w s\Big] = \frac{1}{1-s}\mathrm{PE}\Big[[1,0,\ldots,0]_x[1]_w t + (w^2+w^{-2})s\Big]\,. \tag{4.4}$$

The $\frac{1}{1-s}$ factor above completely decouples from the Molien integral and corresponds to the $\mathbb{C}$ factor of the master space and vortex moduli space. It relates to the center of mass position of the 2 vortices and can be ignored in the following discussion of moduli spaces.

We call the vortex master and moduli spaces reduced when we factor out the center of mass position contribution and denote the spaces respectively by $\widetilde{\mathcal{F}^\flat}_{2,N}$ and $\widetilde{\mathcal{V}}_{2,N}$. The partial $\mathbb{C}^*$ quotient of the master space does not involve the center of mass $\mathbb{C}$ factor. The Hilbert series of the reduced moduli space $\widetilde{\mathcal{V}}_{2,N}$ is obtained therefore simply from the full moduli space $\mathcal{V}_{2,N}$ Hilbert series

$$g(t,s,x;\widetilde{\mathcal{V}}_{2,N}) = (1-s)\times g(t,s,x;\mathcal{V}_{2,N})\,. \tag{4.5}$$

The Hilbert series as a rational function. The first few examples of the refined Hilbert series for the 2 U(N) vortex master spaces are

$$\begin{aligned}
g(t,s,x;\widetilde{\mathcal{F}^\flat}_{2,1}) &= \frac{1}{(1-s^2)(1-st^2)}\,,\\
g(t,s,x;\widetilde{\mathcal{F}^\flat}_{2,2}) &= \frac{1-s^2t^4}{(1-s^2)(1-t^2)(1-x^2st^2)(1-st^2)(1-x^{-2}st^2)}\,,\\
g(t,s,x;\widetilde{\mathcal{F}^\flat}_{2,3}) &= \\
&\frac{1+[0,1]_x st^2-[1,0]_x st^4-s^2t^6}{(1-s^2)(1-x_1^{-1}t^2)(1-x_1^2st^2)(1-x_1x_2^{-1}t^2)(1-x_2t^2)(1-x_2^{-2}st^2)(1-x_1^{-2}x_2^2st^2)}\,.
\end{aligned} \tag{4.6}$$

Quiver fields and convention. $\mathrm{SU}(N)_x$ flavor indices are given by small $i_1, i_2 = 1,\ldots,N$ and $\mathrm{SU}(k)_w$ color indices are given by $\alpha,\beta = 1,2$. The antisymmetric tensor ϵ^{ab} is used as a raising and lowering operator for gauge indices. The Casimir operators are given by,

$$\begin{aligned}
u_1 &= \mathrm{Tr}(\phi) = 0\\
u_2 &= \mathrm{Tr}(\phi^2) = 2\phi_{12}^2 - 2\phi_{11}\phi_{22}\,.
\end{aligned} \tag{4.7}$$

4.1 2 U(1) vortices on $\mathbb{C}$

The Hilbert series of the master space of 2 U(1) vortices is given by the following Molien integral,

$$g(t,s,x;\mathcal{F}^\flat_{2,1}) = \oint \mathrm{d}\mu_{\mathrm{SU}(2)}\mathrm{PE}\Big[[1]_w t + [2]_w s\Big]\,, \tag{4.8}$$

where $\mathrm{d}\mu_{\mathrm{SU}(2)}$ is the SU(2) Haar measure. The Hilbert series is

$$g(t,s,x;\widetilde{\mathcal{F}^\flat}_{2,1}) = \frac{1}{(1-s^2)(1-st^2)}\,. \tag{4.9}$$

The plethystic logarithm is

$$\mathrm{PL}\Big[g(t,s,x;\widetilde{\mathcal{F}^\flat}_{2,1})\Big] = s^2 + st^2\,. \tag{4.10}$$

We note that $\widetilde{\mathcal{F}^\flat}_{2,1}$ has dimension 2. The generators are

$$\begin{aligned}
s^2 &\to u_2 = \mathrm{Tr}(\phi^2)\\
st^2 &\to A = \epsilon^{\alpha_1\alpha_2}\epsilon^{\beta_1\beta_2}Q_{\alpha_1}\phi_{\alpha_2\beta_1}Q_{\beta_2}\,.
\end{aligned} \tag{4.11}$$

$\widetilde{\mathcal{F}^\flat}_{2,1}$ is a freely generated space where the generators do not form any relations.

Vortex moduli space. The $\mathbb{C}^*$ projection of the master space gives

$$\widetilde{\mathcal{V}}_{2,1} = \widetilde{\mathcal{F}^\flat}_{2,1}/\{A \simeq \lambda^2 A\} = \mathbb{C}\,. \tag{4.12}$$

For the general case of k U(1) vortices, the moduli space is simply

$$\widetilde{\mathcal{V}}_{k,1} = \widetilde{\mathcal{F}^\flat}_{k,1}/\{A \simeq \lambda^2 A\} = \mathbb{C}^{k-1}\,. \tag{4.13}$$

The metric for the 2 U(1) vortex moduli space was studied in [8, 32].

4.2 2 U(2) vortices on $\mathbb{C}$

The 2 U(2) vortex master space has the following Molien integral for the Hilbert series,

$$g(t,s,x;\mathcal{F}^\flat_{2,2}) = \oint d\mu_{\mathrm{SU}(2)}\mathrm{PE}\Big[[1]_w[1]_x t + [2]_w s\Big]\,, \tag{4.14}$$

where $[1]_x$ is the character of the fundamental representation of SU(2). By removing the contribution from the center of mass position of the 2 vortices, the Hilbert series is

$$g(t,s,x;\widetilde{\mathcal{F}^\flat}_{2,2}) = \frac{1-s^2t^4}{(1-s^2)(1-t^2)(1-x^2st^2)(1-st^2)(1-x^{-2}st^2)}\,. \tag{4.15}$$

We observe that $\widetilde{\mathcal{F}^\flat}_{2,2}$ is a complete intersection of dimension 4. The character expansion of the Hilbert series is

$$g(t,s,x;\widetilde{\mathcal{F}^\flat}_{2,2}) = \frac{1}{1-s^2}\sum_{n_0,n_1=0}^{\infty}[2n_1]_x s^{n_1}t^{2n_0+2n_1}\,. \tag{4.16}$$

The plethystic logarithm of the Hilbert series takes the form

$$\mathrm{PL}\Big[g(t,s,x;\widetilde{\mathcal{F}^\flat}_{2,2})\Big] = s^2+t^2+[2]_x st^2 - s^2t^4\,. \tag{4.17}$$

The generators of the master space are

$$\begin{aligned} s^2 &\rightarrow u_2 = \mathrm{Tr}(\phi^2) \\ t^2 &\rightarrow M = \epsilon^{\alpha_1\alpha_2}\epsilon_{ij}Q^i_{\alpha_1}Q^j_{\alpha_2} \\ [2]_x st^2 &\rightarrow A^{ij} = \epsilon^{\alpha_1\alpha_2}\epsilon^{\beta_1\beta_2}Q^i_{\alpha_1}\phi_{\alpha_2\beta_1}Q^j_{\beta_2}\,, \end{aligned} \tag{4.18}$$

where we note that

$$A^{ij} = A^{ji}\,. \tag{4.19}$$

Quadratic relations. Let us consider the symmetric product

$$\mathrm{Sym}^2\;[2]_x = [4]_x + [0]_x\,, \tag{4.20}$$

and the term in the plethystic logarithm corresponding to the quadratic relation,

$$-s^2t^4\,. \tag{4.21}$$

The quadratic relation can be constructed as follows,

$$-s^2t^4 \rightarrow R = \det A - \frac{1}{8}u_2M^2 = 0\,. \tag{4.22}$$

Note that $-s^2t^4$ is the corresponding contribution in the plethystic logarithm of the Hilbert series shown in (4.17).

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Vortex moduli space. The $\mathbb{C}^*$ projection of $\widetilde{\mathcal{F}^\flat}_{2,2}$ depends on the U(1) gauge charges encoded in the Q^i_α fugacity t in the Hilbert series. The vortex moduli space is

$$\widetilde{\mathcal{V}}_{2,2} = \widetilde{\mathcal{F}^\flat}_{2,2}/\{M \simeq \lambda^2 M, A^{ij} \simeq \lambda^2 A^{ij}\}\,, \tag{4.23}$$

where

$$\widetilde{\mathcal{F}^\flat}_{2,2} = \mathbb{C}[u_2, M, A^{ij}]/\{R = 0\}\,. \tag{4.24}$$

The dimension of $\widetilde{\mathcal{V}}_{2,2}$ is 3. This reproduces the result in [6, 7].

4.3 2 U(3) vortices on $\mathbb{C}$

The Hilbert series for the 2 U(3) vortex master space is given by the Molien integral

$$g(t, s, x; \mathcal{F}^\flat_{2,3}) = \oint \mathrm{d}\mu_{\mathrm{SU}(2)} \mathrm{PE}\Big[[1]_w[1,0]_x + [2]_w s\Big]\,. \tag{4.25}$$

When the contribution from the center of mass of the vortices is removed, the integral gives

$$g(t,s,x;\widetilde{\mathcal{F}^\flat}_{2,3}) = \frac{1+[0,1]_x st^2-[1,0]_x st^4-s^2t^6}{(1-s^2)(1-x_1^{-1}t^2)(1-x_1^2 st^2)(1-x_1x_2^{-1}t^2)(1-x_2t^2)(1-x_2^{-2}st^2)(1-x_1^{-2}x_2^2 st^2)}\,, \tag{4.26}$$

where we notice that $\widetilde{\mathcal{F}^\flat}_{2,3}$ is a Calabi-Yau space of dimension 6.[5] The character expansion is

$$g(t,s,x;\widetilde{\mathcal{F}^\flat}_{2,3}) = \frac{1}{1-s^2}\sum_{n_0,n_1=0}^{\infty} [2n_1, n_0]_x s^{n_1} t^{2n_0+2n_1}\,. \tag{4.27}$$

The plethystic logarithm of the Hilbert series has the following expansion

$$\mathrm{PL}\Big[g(t,s,x;\widetilde{\mathcal{F}^\flat}_{2,3})\Big] = s^2 + [0,1]_x t^2 + [2,0]_x st^2 - [1,0]_x st^4 - [0,2]_x s^2t^4 + \dots\,. \tag{4.28}$$

The expansion is infinite and therefore we identify the master space as being a non-complete intersection. The generators can be identified from the above plethystic logarithm as

$$\begin{aligned} s^2 &\to u_2 = \mathrm{Tr}(\phi^2) \\ [0,1]_x t^2 &\to M^{ij} = \epsilon^{\alpha_1\alpha_2} Q^i_{\alpha_1} Q^j_{\alpha_2} \\ [2,0]_x st^2 &\to A^{ij} = \epsilon^{\alpha_1\alpha_2}\epsilon^{\beta_1\beta_2} Q^i_{\alpha_1}\phi_{\alpha_2\beta_1} Q^j_{\beta_2}\,, \end{aligned} \tag{4.29}$$

where we have

$$M^{ij} = -M^{ji}\,,\ \ A^{ij} = A^{ji}\,. \tag{4.30}$$

[5] Under a theorem by Stanley [33], when the numerator of the Hilbert series as rational function is palindromic, the corresponding moduli space is Calabi-Yau.

Quadratic relations. For the quadratic relations,[6] we consider the following products of representations,

$$\begin{aligned} [0,1]_x \times [2,0]_x &= [2,1]_x + [1,0]_x \,, \\ \mathrm{Sym}^2\, [2,0]_x &= [4,0]_x + [0,2]_x \,. \end{aligned} \tag{4.31}$$

From the plethystic logarithm in (4.28), we identify the terms

$$-[1,0]_x s t^4 - [0,2]_x s^2 t^4 \tag{4.32}$$

as corresponding to the following quadratic relations respectively,

$$\begin{aligned} -[1,0]_x s t^4 &\to R_i = \frac{1}{2}\epsilon^{jkl} A_{ij} M_{kl} = 0 \\ -[0,2]_x s^2 t^4 &\to S_{ij} = A_{ii}A_{jj} - A_{ij}^2 + \frac{1}{2} u_2 M_{ij} M_{ji} = 0 \,. \end{aligned} \tag{4.33}$$

where

$$S_{[ij]} = 0 \,. \tag{4.34}$$

Vortex moduli space. The vortex moduli space is

$$\widetilde{\mathcal{V}}_{2,3} = \widetilde{\mathcal{F}^\flat}_{2,3} / \{ M_k \simeq \lambda^2 M_k, A^{ij} \simeq \lambda^2 A^{ij} \} \,, \tag{4.35}$$

where

$$\widetilde{\mathcal{F}^\flat}_{2,3} = \mathbb{C}[u_2, M_k, A^{ij}] / \{ R_i = 0, S_{ij} = 0 \} \,. \tag{4.36}$$

The dimension of the above reduced vortex moduli space is 5. The 2 U(2) vortex moduli space has been studied in [5] in the context of cosmic U(2) strings.

4.4 2 U(4) vortices on $\mathbb{C}$

The master space of the 2 U(4) vortex theory has the Molien integral for the Hilbert series,

$$g(t,s,x;\mathcal{F}^\flat_{2,4}) = \oint \mathrm{d}\mu_{\mathrm{SU}(2)}\, \mathrm{PE}\Big[[1]_w [1,0,0]_x t + [2]_w s \Big] \,, \tag{4.37}$$

where $[1,0,0]_x$ is the character for the fundamental representation of $\mathrm{SU}(4)_x$.

The character expansion is

$$g(t,s,x;\widetilde{\mathcal{F}^\flat}_{2,4}) = \frac{1}{1-s^2} \sum_{n_0,n_1=0}^{\infty} [2n_1, n_0, 0]_x s^{n_1} t^{2n_0+2n_1} \,. \tag{4.38}$$

[6]Relations among relations, also known as *syzygies*, are derived from the quadratic relations which we present here. A simple example illustrating this is the Wess-Zumino model with 3 chiral multiplets X, Y and Z and the superpotential $W = XYZ$. The reader is encouraged to review [24] for a pedagogical introduction.

The corresponding plethystic logarithm is given by

$$\begin{aligned}\mathrm{PL}\left[g(t,s,x;\widetilde{\mathcal{F}^\flat}_{2,4})\right] &= s^2+[0,1,0]_x t^2+[2,0,0]_x st^2-t^4-[1,0,1]_x st^4\\ &\quad -[0,2,0]_x s^2t^4+\dots\,. \end{aligned} \tag{4.39}$$

The plethystic logarithm has an infinite expansion. Accordingly, the reduced master space is a non-complete intersection Calabi-Yau space of dimension 8.

The generators are

$$\begin{aligned} s^2 &\to u_2=\mathrm{Tr}(\phi^2)\\ [0,1,0]_x t^2 &\to M^{ij}=\epsilon^{\alpha_1\alpha_2}Q^i_{\alpha_1}Q^j_{\alpha_2}\\ [2,0,0]_x st^2 &\to A^{ij}=Q^i_{\alpha_1}\phi^{\alpha_1\alpha_2}Q^j_{\alpha_2}\,, \end{aligned} \tag{4.40}$$

where

$$M^{ij}=-M^{ji}\,,\ A^{ij}=A^{ji}\,. \tag{4.41}$$

Quadratic relations. The terms in the plethystic logarithm in (4.39) corresponding to the quadratic relations are

$$-t^4-[1,0,1]_x st^4-[0,2,0]_x s^2t^4\,. \tag{4.42}$$

Let us consider the following products of SU(4) representations

$$\begin{aligned} \mathrm{Sym}^2\ [0,1,0]_x &= [0,2,0]_x+[0,0,0]_x\,,\\ [2,0,0]_x\times[0,1,0]_x &= [2,1,0]_x+[1,0,1]_x\,,\\ \mathrm{Sym}^2\ [2,0,0]_x &= [4,0,0]_x+[0,2,0]_x\,. \end{aligned} \tag{4.43}$$

Using the representation products, we construct the quadratic relations as follows

$$\begin{aligned} -t^4 &\to R=\frac{1}{8}\epsilon^{ijkl}M_{ij}M_{kl}=0\\ -[1,0,1]_x st^4 &\to S^i{}_j=\frac{1}{2}\epsilon^{iklm}A_{jk}M_{lm}=0\\ -[0,2,0]_x s^2t^4 &\to T_{ijkl}=A_{ik}A_{jl}-A_{il}A_{jk}+\frac{1}{2}u_2M_{ij}M_{kl}=0\,. \end{aligned} \tag{4.44}$$

The relations exhibit the following properties,

$$\begin{aligned} &S^i{}_i=0\\ &T_{ijkl}=T_{jilk}\,,T_{ijkl}=-T_{jikl}\,,T_{ijkl}=-T_{ijlk}\ \ . \end{aligned} \tag{4.45}$$

Vortex moduli space. We can express the vortex moduli space as the following $\mathbb{C}^*$ projection

$$\widetilde{\mathcal{V}}_{2,4}=\widetilde{\mathcal{F}^\flat}_{2,4}/\{M^{ij}\simeq\lambda^2M^{ij},A^{ij}\simeq\lambda^2A^{ij}\}\,, \tag{4.46}$$

where λ is the $\mathbb{C}^*$ parameter. The master space is given by

$$\widetilde{\mathcal{F}^\flat}_{2,4}=\mathbb{C}[u_2,M^{ij},A^{ij}]/\{R=0,S^i{}_j=0,T_{ijkl}=0\}\,. \tag{4.47}$$

The dimension of the above reduced vortex moduli space is 7.

4.5 2 U(5) vortices on $\mathbb{C}$

Let us consider the master space of 2 U(5) vortices whose Hilbert series can be computed via the following Molien integral,

$$g(t,s,x;\mathcal{F}^\flat_{2,5}) = \oint \mathrm{d}\mu_{\mathrm{SU}(2)}\mathrm{PE}\Big[[1]_w[1,0,0,0]_x + [2]_w s\Big]\,. \tag{4.48}$$

The character expansion of the Hilbert series is

$$g(t,s,x;\widetilde{\mathcal{F}^\flat}_{2,5}) = \frac{1}{1-s^2}\sum_{n_0,n_1=1}^{\infty}[2n_1,n_0,0,0]_x s^{n_1}t^{2n_0+2n_1}\,. \tag{4.49}$$

The plethystic logarithm of the Hilbert series is

$$\begin{aligned}\mathrm{PL}\Big[g(t,s,x;\widetilde{\mathcal{F}^\flat}_{2,3})\Big] &= s^2+[0,1,0,0]_x t^2+[2,0,0,0]_x st^2-[0,0,0,1]_x t^4\\ &\quad -[1,0,1,0]_x st^4-[0,2,0,0]_x s^2t^4+\dots\,.\end{aligned} \tag{4.50}$$

The reduced master space is a non-complete intersection Calabi-Yau and has dimension 10.

The generators of the master space are

$$\begin{aligned} s^2 &\to u_2=\mathrm{Tr}(\phi^2)\\ [0,1,0,0]_x t^2 &\to M^{ij}=\epsilon^{\alpha_1\alpha_2}Q^i_{\alpha_1}Q^j_{\alpha_2}\\ [2,0,0,0]_x st^2 &\to A^{ij}=Q^i_{\alpha_1}\phi^{\alpha_1\alpha_2}Q^j_{\alpha_2}\,,\end{aligned} \tag{4.51}$$

where we have

$$M^{ij}=-M^{ji}\,,\ A^{ij}=A^{ji}\,. \tag{4.52}$$

Quadratic relations. The plethystic logarithm of the Hilbert series in (4.50) has the following terms corresponding to the quadratic relations between generators,

$$-[0,0,0,1]_x t^4-[1,0,1,0]_x st^4-[0,2,0,0]_x s^2t^4\,. \tag{4.53}$$

Let us consider the following representation products

$$\begin{aligned}\mathrm{Sym}^2[0,1,0,0]_x &= [0,2,0,0]_x+[0,0,0,1]_x\,,\\ [2,0,0,0]_x\times[0,1,0,0]_x &= [2,1,0,0]_x+[1,0,1,0]_x\,,\\ \mathrm{Sym}^2[2,0,0,0]_x &= [4,0,0,0]_x+[0,2,0,0]_x\,.\end{aligned} \tag{4.54}$$

Using the representation products, we construct the quadratic relations as follows,

$$\begin{aligned} -[0,0,0,1]_x t^4 &\to R^i=\frac{1}{8}\epsilon^{ijklm}M_{jk}M_{lm}=0\\ -[1,0,1,0]_x st^4 &\to {S^{ij}}_k=\frac{1}{2}\epsilon^{ijlmn}A_{kl}M_{mn}=0\\ -[0,2,0,0]_x s^2t^4 &\to T_{ijkl}=A_{ik}A_{jl}-A_{il}A_{jk}+\frac{1}{2}u_2M_{ij}M_{kl}=0\,,\end{aligned} \tag{4.55}$$

where the relations satisfy

$$\begin{aligned} &{S^{ik}}_k=0\,,\\ &T_{ijkl}=T_{jilk}\,,T_{ijkl}=-T_{jikl}\,,T_{ijkl}=-T_{ijlk}\,.\end{aligned} \tag{4.56}$$

Vortex moduli space. Given the generators and the quadratic relations of the master space, we can express the vortex moduli space as the following $\mathbb{C}^*$ projection,

$$\widetilde{\mathcal{V}}_{2,5} = \widetilde{\mathcal{F}^\flat}_{2,5}/\{M^{ij} \simeq \lambda^2 M^{ij}, A^{ij} \simeq \lambda^2 A^{ij}\}\,, \tag{4.57}$$

where λ is the $\mathbb{C}^*$ parameter. The master space is

$$\widetilde{\mathcal{F}^\flat}_{2,5} = \mathbb{C}[u_2, M^{ij}, A^{ij}]/\{R^i = 0, {S^{ij}}_k = 0, T_{ijkl} = 0\}\,. \tag{4.58}$$

The dimension of $\widetilde{\mathcal{F}^\flat}_{2,5}$ is 9.

4.6 2 U(N) vortices on $\mathbb{C}$

For the general case of 2 U(N) vortices, the Molien integral for the Hilbert series of the master space is

$$g(t,s,x;\mathcal{F}^\flat_{2,N}) = \oint \mathrm{d}\mu_{\mathrm{SU}(2)}\mathrm{PE}\Big[[1]_w[1,0,\ldots,0]_x + [2]_w s\Big]\,, \tag{4.59}$$

where $[1,0,\ldots,0]_x$ is the character for the fundamental representation of $\mathrm{SU}(N)_x$. When unrefined by setting the fugacities for the $\mathrm{SU}(N)_x$ characters to $x_i = 1$, the Hilbert series for the first few values of N are

$$\begin{aligned}
g(t,s;\widetilde{\mathcal{F}^\flat}_{2,1}) &= \frac{1}{(1-s^2)(1-st^2)}\,,\\
g(t,s;\widetilde{\mathcal{F}^\flat}_{2,2}) &= \frac{1-s^2t^4}{(1-s^2)(1-t^2)(1-st^2)^2}\,,\\
g(t,s;\widetilde{\mathcal{F}^\flat}_{2,3}) &= \frac{1+3st^2-3st^4-s^2t^6}{(1-s^2)(1-t^2)^3(1-st^2)^3}\,,\\
g(t,s;\widetilde{\mathcal{F}^\flat}_{2,4}) &= \frac{1+t^2+6st^2-9st^4+s^2t^4+st^6-9s^2t^6+6s^2t^8+s^3t^8+s^3t^{10}}{(1-s^2)(1-t^2)^5(1-st^2)^4}\,,\\
g(t,s;\widetilde{\mathcal{F}^\flat}_{2,5}) &= \frac{1}{(1-s^2)(1-t^2)^7(1-st^2)^5}\times(1+3t^2+10st^2+t^4-15st^4\\
&\quad +5s^2t^4-5st^6-40s^2t^6+40s^2t^8+5s^3t^8-5s^2t^{10}+15s^3t^{10}\\
&\quad -s^4t^{10}-10s^3t^{12}-3s^4t^{12}-s^4t^{14})\,,\\
g(t,s;\widetilde{\mathcal{F}^\flat}_{2,6}) &= \frac{1}{(1-s^2)(1-t^2)^9(1-st^2)^6}\times(1+6t^2+15st^2+6t^4-15st^4\\
&\quad +15s^2t^4+t^6-36st^6-120s^2t^6+s^3t^6-6st^8+126s^2t^8+6s^3t^8\\
&\quad +6s^2t^{10}+126s^3t^{10}-6s^4t^{10}+s^2t^{12}-120s^3t^{12}-36s^4t^{12}+s^5t^{12}\\
&\quad +15s^3t^{14}-15s^4t^{14}+6s^5t^{14}+15s^4t^{16}+6s^5t^{16}+s^5t^{18})\,.\\
g(t,s;\widetilde{\mathcal{F}^\flat}_{2,7}) &= \frac{1}{(1-s^2)(1-t^2)^{11}(1-st^2)^7}\times(1+10t^2+21st^2+20t^4\\
&\quad +35s^2t^4+10t^6-112st^6-280s^2t^6+7s^3t^6+t^8-70st^8+224s^2t^8\\
&\quad -35s^3t^8-7st^{10}+210s^2t^{10}+658s^3t^{10}-21s^4t^{10}+21s^2t^{12}-658s^3t^{12}\\
&\quad -210s^4t^{12}+7s^5t^{12}+35s^3t^{14}-224s^4t^{14}+70s^5t^{14}-s^6t^{14}-7s^3t^{16}\\
&\quad +280s^4t^{16}+112s^5t^{16}-10s^6t^{16}-35s^4t^{18}-20s^6t^{18}-21s^5t^{20}\\
&\quad -10s^6t^{20}-s^6t^{22})\,,
\end{aligned} \tag{4.60}$$

where we have removed the contribution from the centre of mass position of the 2 vortices.

We observe that the numerators in (4.60) are all palindromic. This indicates that the vortex master space is a Calabi-Yau manifold. Refined and as a character expansion, the Hilbert series for the reduced 2 U(N) vortex master space takes the following general form

$$g(t,s,x;\widetilde{\mathcal{F}^\flat}_{2,N}) = \frac{1}{1-s^2}\sum_{n_0=0}^{\infty}\sum_{n_1=0}^{\infty}[2n_1,n_0,0,\dots,0]_x s^{n_1}t^{2(n_0+n_1)}\,. \tag{4.61}$$

The plethystic logarithm of the Hilbert series is

$$\begin{aligned}\mathrm{PL}\Big[g(t,s,x;\widetilde{\mathcal{F}^\flat}_{2,N})\Big] &= s+s^2+[0,1,0,\dots,0]_x t^2+[2,0,\dots,0]_x st^2\\ &\quad -[0,0,0,1,0,\dots,0]_x t^4-[1,0,1,0,\dots,0]_x st^4\\ &\quad -[0,2,0,\dots,0]_x s^2t^4+\dots\,.\end{aligned} \tag{4.62}$$

The first positive terms of the plethystic logarithm correspond to the generators

$$\begin{aligned} s^2 &\to u_2=\mathrm{Tr}(\phi^2)\\ [0,1,0,\dots,0]_x t^2 &\to M^{ij}=\epsilon^{\alpha_1\alpha_2}Q^i_{\alpha_1}Q^j_{\alpha_2}\\ [2,0,\dots,0]_x st^2 &\to A^{ij}=Q^i_{\alpha_1}\phi^{\alpha_1\alpha_2}Q^j_{\alpha_2}\,.\end{aligned} \tag{4.63}$$

The generators satisfy the following,

$$M^{ij}=-M^{ji}\,,A^{ij}=A^{ij}\,. \tag{4.64}$$

Quadratic relations. The plethystic logarithm of the Hilbert series of the master space exhibits the following terms corresponding quadratic relations between the generators of the master space,

$$-[0,0,0,1,0,\dots,0]_x t^4-[1,0,1,0\dots,0]_x st^2-[0,2,0,\dots,0]_x s^2t^4\,. \tag{4.65}$$

which respectively correspond to the following quadratic relations between generators,

$$\begin{aligned} -[0,0,0,1,0,\dots,0]_x t^4 &\to R^{i_1\dots i_{N-4}}=\frac{1}{8}\epsilon^{i_1\dots i_{N-4}jklm}M_{jk}M_{lm}=0\\ -[1,0,1,0,\dots,0]_x st^2 &\to S^{i_1\dots i_{N-3}}{}_j=\frac{1}{2}\epsilon^{i_1\dots i_{N-3}klm}A_{jk}M_{lm}=0\\ -[0,2,0,\dots,0]_x s^2t^4 &\to T_{ijkl}=A_{ik}A_{jl}-A_{il}A_{jk}+\frac{1}{2}u_2M_{ij}M_{kl}=0\,.\end{aligned} \tag{4.66}$$

The above relations satisfy

$$\begin{aligned} S^{i_1\dots i_{N-3}}{}_{i_{N-3}} &=0\,,\\ T_{ijkl}=T_{jilk}\,,\quad T_{ijkl}&=-T_{jikl}\,,\quad T_{ijkl}=-T_{ijlk}\,.\end{aligned} \tag{4.67}$$

Vortex moduli space. The moduli space of 2 U(N) vortices is expressed as the following $\mathbb{C}^*$ projection,

$$\widetilde{\mathcal{V}}_{2,N}=\widetilde{\mathcal{F}^\flat}_{2,N}/\{M^{ij}\simeq\lambda^2M^{ij},A^{ij}\simeq\lambda^2A^{ij}\}\,, \tag{4.68}$$

where the master space is given by

$$\widetilde{\mathcal{F}^\flat}_{2,N}=\mathbb{C}[u_2,M^{ij},A^{ij}]/\{R^{i_1\dots i_{N-4}}=0,S^{i_1\dots i_{N-3}}{}_j=0,T_{ijkl}=0\}\,. \tag{4.69}$$

The dimension of the reduced 2 U(N) vortex moduli space is $2N-1$.

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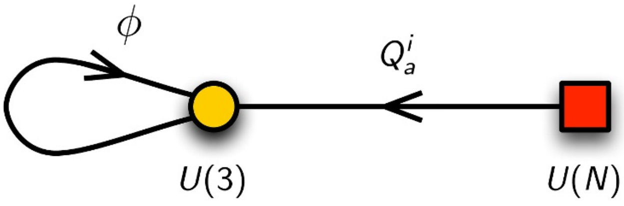

Figure 7. Quiver diagram of the 3 U(N) vortex theory.

5 3 U(N) vortices on $\mathbb{C}$

The quiver diagram of the 3 U(N) vortex theory is shown in figure 7.

The moduli space. The moduli space of 3 U(N) vortices $\mathcal{V}_{3,N}$ can be expressed as a $\mathbb{C}^*$ projection of the master space $\mathcal{F}^\flat_{3,N}$ of the vortex theory. Given the generators of the master spaces, $x_1, \ldots, x_d$, the $\mathbb{C}^*$ projection acts as follows on the master space coordinates

$$(x_1, \ldots, x_d) \simeq (\lambda^{w_1} x_1, \ldots, \lambda^{w_d} x_d)\,. \tag{5.1}$$

λ is the $\mathbb{C}^*$ parameter and $w_1, \ldots, w_d$ are the U(1) weights for the $\mathbb{C}^*$ projection of the master space. Following the coordinate identification in (5.1), the master space is $\mathbb{C}^*$ projected as follows,

$$\mathcal{V}_{3,N} = \mathcal{F}^\flat_{3,N} / \{x_1 \simeq \lambda^{w_1} x_1, \ldots, x_d \simeq \lambda^{w_d} x_d\}\,. \tag{5.2}$$

The Molien integral and Hilbert series. The Hilbert series of the master space of 3 U(N) vortices is given by the following Molien integral

$$g(t, s, x; \mathcal{F}^\flat_{3,N}) = \oint \mathrm{d}\mu_{\mathrm{SU}(3)}\ \mathrm{PE}\Big[[1, 0, \ldots, 0]_x [0, 1]_w t + [1, 1]_w s\Big]\,, \tag{5.3}$$

where $\mathrm{d}\mu_{\mathrm{SU}(3)}$ is the Haar measure for SU(3). The entries of the plethystic exponential correspond as expected to Q^i_α transforming in $[1, 0, \ldots, 0]_x [0, 1]_w t$ and ϕ transforming in $[1, 1]_w s$.

Center of mass position. The integrand in the Molien integral for the Hilbert series of the vortex master space can be rewritten as follows,

$$\begin{aligned} \mathrm{PE}\Big[[1, 0, \ldots, 0]_x [0, 1]_w t + [1, 1]_w s\Big] = \frac{1}{1-s} \mathrm{PE}\Big[&[1, 0, \ldots, 0]_x [0, 1]_w t \\ &+ (w_1 w_2 + w_1^2 w_2^{-1} + w_1^{-1} w_2^2 + 1 + w_1 w_2^{-2} + w_1^{-2} w_2 + w_1^{-1} w_2^{-1})\Big]\,, \end{aligned} \tag{5.4}$$

where the character of the adjoint of SU(3) is given by

$$[1, 1]_w = w_1 w_2 + w_1^2 w_2^{-1} + w_1^{-1} w_2^2 + 2 + w_1 w_2^{-2} + w_1^{-2} w_2 + w_1^{-1} w_2^{-1}\,. \tag{5.5}$$

The prefactor $\frac{1}{1-s}$ in (5.4) refers to the center of mass position of the 3 vortices. The center of mass position contributes in the vortex master space $\mathcal{F}^\flat_{3,N}$ and moduli space $\mathcal{V}_{3,N}$ with a $\mathbb{C}$ factor. This contribution does not interact with the $\mathbb{C}^*$ projection of the master space and therefore can be safely taken out from the following discussion. The reduced vortex master and moduli spaces are denoted respectively by $\widetilde{\mathcal{F}^\flat}_{3,N}$ and $\widetilde{\mathcal{V}}_{3,N}$.

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Quiver fields and convention. The SU(N) flavor indices are given by $i_1, i_2, i_3 = 1, \ldots, N$ and SU(k) color indices are given by $\alpha, \beta = 1, 2, 3$. The components for the adjoint $\phi_{\alpha\beta}$ satisfy $\phi_{33} = -\phi_{11} - \phi_{22}$ such that

$$\phi_\alpha{}^\alpha = 0\,. \tag{5.6}$$

Note that for SU(3) we use $\delta_{\alpha\beta}$ as a raising or lowering operator. Using the choice of adjoint above, we obtain the following traces which we use in the discussion of vortex moduli spaces,

$$\begin{aligned} u_1 &= \mathrm{Tr}(\phi) = 0 \\ u_2 &= \mathrm{Tr}(\phi^2) = 2\phi_{11}^2 + 2\phi_{12}\phi_{21} + 2\phi_{11}\phi_{22} + 2\phi_{22}^2 + 2\phi_{13}\phi_{31} + 2\phi_{23}\phi_{32} \\ u_3 &= \mathrm{Tr}(\phi^3) = 3\phi_{11}\phi_{12}\phi_{21} - 3\phi_{11}^2\phi_{22} + 3\phi_{12}\phi_{21}\phi_{22} - 3\phi_{11}\phi_{22}^2 - 3\phi_{13}\phi_{22}\phi_{31} + 3\phi_{12}\phi_{23}\phi_{31} \\ &\qquad\quad +3\phi_{13}\phi_{21}\phi_{32} - 3\phi_{11}\phi_{23}\phi_{32}\,. \end{aligned} \tag{5.7}$$

5.1 3 U(1) vortices on $\mathbb{C}$

The Hilbert series of the 3 U(1) vortex master space is

$$g(t, s, x; \widetilde{\mathcal{F}^\flat}_{3,1}) = \frac{1}{(1 - s^2)(1 - s^3)(1 - s^3 t^3)}\,. \tag{5.8}$$

The master space generators are

$$\begin{aligned} s^2 &\to u_2 = \mathrm{Tr}(\phi^2) \\ s^3 &\to u_3 = \mathrm{Tr}(\phi^3) \\ s^3 t^3 &\to A_{012} = \epsilon^{\alpha_1\alpha_2\alpha_3} Q_{\alpha_1} \phi_{\alpha_2}^{\beta_1} Q_{\beta_1} \phi_{\alpha_3}^{\beta_2} \phi_{\beta_2}^{\beta_3} Q_{\beta_3}\,, \end{aligned} \tag{5.9}$$

and the vortex moduli space under the $\mathbb{C}^*$ projection becomes

$$\widetilde{\mathcal{V}}_{3,1} = \widetilde{\mathcal{F}^\flat}_{3,1} / \{A_{012} \simeq \lambda^3 A_{012}\} = \mathbb{C}^2\,. \tag{5.10}$$

This agrees with the generalization for k U(1) vortices.

5.2 3 U(2) vortices on $\mathbb{C}$

The Hilbert series of the 3 U(2) vortex master space can be obtained by solving the following Molien integral

$$g(t, s, x; \widetilde{\mathcal{F}^\flat}_{3,2}) = \oint \mathrm{d}\mu_{\mathrm{SU}(3)} \mathrm{PE}\Big[[0,1]_w [1]_x t + [1,1]_w s\Big]\,. \tag{5.11}$$

The solution to the integral is

$$g(t, s, x; \widetilde{\mathcal{F}^\flat}_{3,2}) = \frac{1 + [1]_x s^2 t^3 + [1]_x s^3 t^3 + s^5 t^6}{(1-s^2)(1-s^3)(1-x^3 s^3 t^3)(1-x s t^3)(1-x^{-1} s t^3)(1-x^{-3} s^3 t^3)}\,. \tag{5.12}$$

$\widetilde{\mathcal{F}^\flat}_{3,2}$ is a non-complete intersection Calabi-Yau space of dimension 7.

As a character expansion, the Hilbert series is

$$
\begin{aligned}
g(t,s,x;\widetilde{\mathcal{F}^\flat}_{3,2}) = & \frac{1}{(1-s^2)(1-s^3)} \\
& \times \sum_{n_1=0}^{\infty}\sum_{n_2=0}^{\infty}\sum_{n_3=0}^{\infty}\Bigg[[n_1+n_2+3n_3]_x s^{n_1+2n_2+3n_3} t^{3n_1+3n_2+3n_3} \\
& \qquad + [n_1+n_2]_x s^{n_1+2n_2+3n_3+3} t^{3n_1+3n_2+6n_3+6}\Bigg] . \qquad (5.13)
\end{aligned}
$$

The plethystic logarithm of the Hilbert series is

$$
\begin{aligned}
\mathrm{PL}[g(t,s,x;\mathcal{F}^\flat_{3,2})] = & \; s + s^2 + s^3 + [1]_x s t^3 + [1]_x s^2 t^3 + [3]_x s^3 t^3 \\
& -[2]_x s^4 t^6 - [2]_x s^5 t^6 - [2]_x s^6 t^6 + \ldots . \qquad (5.14)
\end{aligned}
$$

The generators of the master space are identified from the plethystic logarithm as follows

$$
\begin{aligned}
s^2 & \rightarrow u_2 = \mathrm{Tr}(\phi^2) \\
s^3 & \rightarrow u_3 = \mathrm{Tr}(\phi^3) \\
[1]_x s t^3 & \rightarrow A^i_{001} = \epsilon^{\alpha_1\alpha_2\alpha_3}\epsilon_{jk} Q^j_{\alpha_1} Q^k_{\alpha_2} \phi^{\beta_1}_{\alpha_3} Q^{i_3}_{\beta_1} \\
[1]_x s^2 t^3 & \rightarrow \begin{cases} A^i_{002} = \epsilon^{\alpha_1\alpha_2\alpha_3}\epsilon_{jk} Q^j_{\alpha_1} Q^k_{\alpha_2} \phi^{\beta_1}_{\alpha_3} \phi^{\beta_2}_{\beta_1} Q^i_{\beta_2} \\ A^i_{011} = \epsilon^{\alpha_1\alpha_2\alpha_3}\epsilon_{jk} Q^i_{\alpha_1} \phi^{\beta_1}_{\alpha_2} Q^j_{\beta_1} \phi^{\beta_2}_{\alpha_3} Q^k_{\beta_2} \\ \rightarrow A^i_{002} = -A^i_{011} \end{cases} \\
[3]_x s^3 t^3 & \rightarrow \begin{cases} A^{ijk}_{012} = \epsilon^{\alpha_1\alpha_2\alpha_3} Q^i_{\alpha_1} \phi^{\beta_1}_{\alpha_2} Q^j_{\beta_1} \phi^{\beta_2}_{\alpha_3} \phi^{\beta_3}_{\beta_2} Q^k_{\beta_3} \\ \rightarrow S^{ijk}_{012} = A^{ijk}_{012} + A^{jki}_{012} + A^{kij}_{012} \end{cases} .
\end{aligned}
$$

The indices of the generators $A^{i_1 i_2 i_3}_{012}$ are symmetrized to obtain S^{ijk}_{012}.

Quadratic relations. The plethystic logarithm of the Hilbert series of the vortex master space encodes the quadratic relations formed by the generators. The following terms of the plethystic logarithm correspond to the quadratic relations,

$$
-[2]_x s^4 t^6 - [2]_x s^5 t^6 - [2]_x s^6 t^6 . \qquad (5.15)
$$

From the above discussion of generators, we recall that the generators are

$$
u_2 \, , u_3 \, , A^{i_1 i_2 i_3}_{002} \, , S^{i_1 i_2 i_3}_{012} . \qquad (5.16)
$$

Let us go through each of the relations separately as follows:

- $-[2]_x s^4 t^6$ *relations.* We note that

$$
\begin{aligned}
\mathrm{Sym}^2[1]_x & = [2]_x \, , \\
[1]_x \times [3]_x & = [4]_x + [2]_x . \qquad (5.17)
\end{aligned}
$$

Accordingly, the relations which transform in $[2]_x s^4 t^6$ can be obtained via the following products of generators which also transform in $[2]_x s^4 t^6$,

$$\begin{aligned} {r_{(I)}}^{ij} &= u_2 A^i_{001} A^j_{001} \,, \\ {r_{(II)}}^{ij} &= A^i_{002} A^j_{002} \,, \\ {r_{(III)}}^{ij} &= \epsilon_{pq} A^p_{001} S^{qij}_{012} \,. \end{aligned} \tag{5.18}$$

Using the above product expressions, the relation at order $[2]_x s^4 t^6$ can be identified as

$$-[2]_x s^4 t^6 \rightarrow R^{ij} = {r_{(I)}}^{ij} - 6{r_{(II)}}^{ij} - 4{r_{(III)}}^{ij} = 0 \,. \tag{5.19}$$

Note that the relation above is symmetric in its indices,

$$R^{ij} = R^{ji} \,. \tag{5.20}$$

- $-[2]_x s^5 t^6$ *relations.* The products that could contribute to a relation at order $[2]_x s^5 t^6$ are as follows,

$$\begin{aligned} {p_{(I)}}^{ij} &= u_3 A^i_{001} A^j_{001} \,, \\ {p_{(II)}}^{ij} &= u_2 A^i_{001} A^j_{002} \,, \\ {p_{(III)}}^{ij} &= \epsilon_{pq} A^p_{002} S^{qij}_{012} \,. \end{aligned} \tag{5.21}$$

Given the above products, we are able to construct the following relation corresponding to the order $[2]_x s^5 t^6$,

$$-[2]_x s^5 t^6 \rightarrow P^{ij} = 2{p_{(I)}}^{ij} - {p_{(II)}}^{ij} - {p_{(II)}}^{ji} - 4{p_{(III)}}^{ij} = 0 \,. \tag{5.22}$$

The above relation is symmetric in its indices,

$$P^{ij} = P^{ji} \,. \tag{5.23}$$

- $-[2]_x s^6 t^6$ *relations.* Let us first write down the products of generators that correspond to order $[2]_x s^6 t^6$ as follows,

$$\begin{aligned} {o_{(I)}}^{ij} &= u_2^2 A^i_{001} A^j_{001} \,, \\ {o_{(II)}}^{ij} &= u_2 A^i_{002} A^j_{002} \,, \\ {o_{(III)}}^{ij} &= u_3 A^i_{001} A^j_{002} \,, \\ {o_{(IV)}}^{ij} &= \epsilon_{pq} A^p_{001} S^{qij}_{012} \,, \\ {o_{(V)}}^{ij} &= \epsilon_{pq} \epsilon_{rs} S^{pri}_{012} S^{qsj}_{012} \,. \end{aligned} \tag{5.24}$$

From the above, the products ${o_{(I)}}^{ij}, {o_{(II)}}^{ij}, {o_{(IV)}}^{ij}$ can be used to form the following relation at order $[2]_x s^6 t^6$,

$$-[2]_x s^6 t^6 \rightarrow O^{ij} = {o_{(I)}}^{ij} - 6{o_{(II)}}^{ij} - 4{o_{(IV)}}^{ij} = 0 \,. \tag{5.25}$$

The relation above is symmetric in its indices,

$$O^{ij} = O^{ji} \,. \tag{5.26}$$

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Vortex moduli space. The vortex moduli space can be expressed as a partial $\mathbb{C}^*$ projection of the vortex master space $\widetilde{\mathcal{F}^\flat}_{3,2}$. The projection is given as follows,

$$\widetilde{\mathcal{V}}_{3,2} = \widetilde{\mathcal{F}^\flat}_{3,2}/\{A_{001}{}^i = \lambda^3 A_{001}{}^i, A_{002}{}^i = \lambda^3 A_{002}{}^i, S_{012}{}^{ijk} = \lambda^3 S_{012}{}^{ijk}\}\,, \tag{5.27}$$

where λ is the $\mathbb{C}^*$ parameter. The vortex moduli space is as expected 5 dimensional. The vortex master space is given by the following quotient

$$\widetilde{\mathcal{F}^\flat}_{3,2} = \mathbb{C}[u_2, u_3, A_{001}{}^i, A_{002}{}^i, S_{012}{}^{ijk}]/\{R^{ij} = 0, P^{ij} = 0, O^{ij} = 0\}\,. \tag{5.28}$$

5.3 3 U(3) vortices on $\mathbb{C}$

For 3 U(3) vortices, the Hilbert series of the master space is given by the following Molien integral

$$g(t, s, x; \widetilde{\mathcal{F}^\flat}_{3,3}) = \oint \mathrm{d}\mu_{\mathrm{SU}(3)}\mathrm{PE}\Big[[0,1]_w[1,0]_x t + [1,1]_w s\Big]\,. \tag{5.29}$$

The Hilbert series for $\widetilde{\mathcal{F}^\flat}_{3,3}$ is

$$\begin{aligned} g(t, s; \widetilde{\mathcal{F}^\flat}_{3,3}) = & \frac{1}{(1-s^2)(1-s^3)(1-t^3)(1-st^3)^4(1-s^3t^3)^3} \times \\ & (1 + 4st^3 + 8s^2t^3 + 7s^3t^3 + s^2t^6 + 5s^3t^6 + 10s^4t^6 + 11s^5t^6 + s^6t^6 \\ & -s^4t^9 - 11s^5t^9 - 10s^6t^9 - 5s^7t^9 - s^8t^9 - 7s^7t^{12} - 8s^8t^{12} - 4s^9t^{12} \\ & -s^{10}t^{15})\,, \end{aligned} \tag{5.30}$$

where for simplicity we have set the global SU(3) fugacities to $x_1 = x_2 = 1$. $\widetilde{\mathcal{F}^\flat}_{3,3}$ is a non-complete intersection Calabi-Yau space of dimension 9. The character expansion of the Hilbert series is

$$\begin{aligned} g(t, s, x; \widetilde{\mathcal{F}^\flat}_{3,3}) = & \frac{1}{(1-s^2)(1-s^3)} \times \\ & \sum_{n_0=0}^{\infty}\sum_{n_1=0}^{\infty}\sum_{n_2=0}^{\infty}\sum_{n_3=0}^{\infty} \Big[[n_1+n_2+3n_3, n_1+n_2]_x s^{n_1+2n_2+3n_3} t^{3n_0+3n_1+3n_2+3n_3} \\ & + [n_1+n_2, n_1+n_2+3n_3+3]_x s^{n_1+2n_2+3n_3+3} t^{3n_0+3n_1+3n_2+6n_3+6} \Big]\,. \end{aligned} \tag{5.31}$$

The plethystic logarithm is

$$\begin{aligned} \mathrm{PL}\Big[g(t, s, x; \widetilde{\mathcal{F}^\flat}_{3,3})\Big] = & \; s^2 + s^3 + t^3 + [1,1]_x st^3 + [1,1]_x s^2t^3 + [3,0]_x s^3t^3 \\ & -(1 + [1,1]_x)s^2t^6 - (1 + 2[1,1]_x + [3,0]_x)s^3t^6 \\ & -(1 + 2[1,1]_x + [2,2]_x + [3,0]_x)s^4t^6 + \ldots\,. \end{aligned} \tag{5.32}$$

The generators of $\widetilde{\mathcal{F}^\flat}_{3,3}$ are as follows

$$
\begin{aligned}
s^2 &\to u_2 = \mathrm{Tr}(\phi^2) \\
s^3 &\to u_3 = \mathrm{Tr}(\phi^3) \\
t^3 &\to \begin{cases} B^{ijk} = \epsilon^{\alpha_1\alpha_2\alpha_3} Q^i_{\alpha_1} Q^j_{\alpha_2} Q^k_{\alpha_3} \\ \to B = B^{123} + B^{231} + B^{312} \end{cases} \\
[1,1]_x s t^3 &\to \begin{cases} {A_{001}}_i^{\ j} = \epsilon^{\alpha_1\alpha_2\alpha_3} \epsilon_{ik_1k_2} Q^{k_1}_{\alpha_1} Q^{k_2}_{\alpha_2} \phi^{\beta}_{\alpha_3} Q^j_{\beta} \\ {A_{001}}_i^{\ i} = 0 \end{cases} \\
[1,1]_x s^2 t^3 &\to \begin{cases} {A_{002}}_i^{\ j} = \epsilon^{\alpha_1\alpha_2\alpha_3} \epsilon_{ik_1k_2} Q^{k_1}_{\alpha_1} Q^{k_2}_{\alpha_2} \phi^{\beta_1}_{\alpha_3} \phi^{\beta_2}_{\beta_1} Q^j_{\beta_2} \\ {A_{011}}_i^{\ j} = \epsilon^{\alpha_1\alpha_2\alpha_3} \epsilon_{ik_1k_2} Q^{j}_{\alpha_1} \phi^{\beta_1}_{\alpha_2} Q^{k_1}_{\beta_1} \phi^{\beta_2}_{\alpha_3} Q^{k_2}_{\beta_2} \\ \to {A_{002}}_i^{\ j} = -{A_{011}}_i^{\ j} \end{cases} \\
[3,0]_x s^3 t^3 &\to \begin{cases} {A_{012}}^{ijk} = \epsilon^{\alpha_1\alpha_2\alpha_3} Q^i_{\alpha_1} \phi^{\beta_1}_{\alpha_2} Q^j_{\beta_1} \phi^{\beta_2}_{\alpha_3} \phi^{\beta_3}_{\beta_2} Q^k_{\beta_3} \\ {S_{012}}^{ijk} = {A_{012}}^{ijk} + {A_{012}}^{jki} + {A_{012}}^{kij} \end{cases} .
\end{aligned}
$$

The generators for $[3,0]_x s^3 t^4$ come only from the partition 012, giving $A_{012}^{i_1 i_2 i_3}$. This in turn is symmetrized to give $S_{012}^{i_1 i_2 i_3}$.

Quadratic relations. The terms in the plethystic logarithm corresponding to quadratic relations are

$$
\begin{gathered}
-s^2t^6 - [1,1]_x s^2 t^6 \\
-s^3t^6 - 2[1,1]_x s^3 t^6 - [3,0]_x s^3 t^6 \\
-s^4t^6 - 2[1,1]_x s^4 t^6 - [2,2]_x s^4 t^6 - [3,0]_x s^4 t^6 \\
-[1,1]_x s^5 t^6 - [2,2]_x s^5 t^6 - [3,0]_x s^5 t^6 \\
-[2,2]_x s^6 t^6 \,.
\end{gathered} \tag{5.33}
$$

We recall, the generators are

$$
u_2 \,, u_3 \,, B \,, {A_{001}}_i^{\ j} \,, {A_{002}}_i^{\ j} \,, {S_{012}}^{ijk} \,. \tag{5.34}
$$

Let us go through each of the quadratic relations that are given by the terms in (5.33) in the plethystic logarithm.

The first set of quadratic relations at orders of s^2t^6 are as follows:

- $-s^2t^6$ *relation.* We see that this quadratic relation transforms as a singlet in $\mathrm{SU}(3)_x$. Considering the following symmetric product which contains the singlet $[0,0]_x$,

$$
\mathrm{Sym}^2\ [1,1]_x = [2,2]_x + [1,1]_x + [0,0]_x \,, \tag{5.35}
$$

we identify the following generator products in order to construct the desired quadratic relation,

$$
\begin{aligned}
{r_{(I)}}_i^{\ j} &= B {A_{002}}_i^{\ j} \\
{r_{(II)}}_i^{\ j} &= {A_{001}}_i^{\ k} {A_{001}}_k^{\ j} \\
r_{(III)} &= u_3 B B \,,
\end{aligned} \tag{5.36}
$$

which all transform in $-s^2t^6$. Using the above products correctly, we construct the relation

$$-s^2t^6 \to R_{(I)} = 4r_{(III)} - 9({r_{(II)}}_1{}^1 + {r_{(II)}}_2{}^2 + {r_{(II)}}_3{}^3) = 0\,. \tag{5.37}$$

- $-[1,1]_x s^2t^6$ *relations.* Given that at this order the relations need to transform in $[1,1]_x s^2t^6$, the following relations can be identified using the products in (5.36)

$$-[1,1]_x s^2t^6 \to {R_{(II)}}_i{}^j = 3{r_{(II)}}_i{}^j - 2{r_{(I)}}_i{}^j = 0\,. \tag{5.38}$$

The next set of quadratic relations are at orders of s^3t^6. In order to construct the quadratic relations, we consider the following representation products,

$$\begin{aligned} \mathrm{Sym}^2[1,1]_x &= [2,2]_x + [1,1]_x + [0,0]_x\,, \\ [1,1]_x \times [1,1]_x &= [2,2]_x + [3,0]_x + [0,3]_x + 2[1,1]_x + [0,0]_x\,. \end{aligned} \tag{5.39}$$

The quadratic relations are as follows:

- $-s^3t^6$ *relation.* For this order, we first consider the following generator products,

$$\begin{aligned} {p_{(I)}}_i{}^j &= {A_{001}}_i{}^k\,{A_{002}}_k{}^j\,, \\ p_{(II)} &= u_3 BB\,. \end{aligned} \tag{5.40}$$

The quadratic relation for this order is

$$P_{(I)} = 4p_{(II)} - 9({p_{(I)}}_1{}^1 + {p_{(I)}}_2{}^2 + {p_{(I)}}_3{}^3) = 0\,. \tag{5.41}$$

- $-[1,1]_x s^3t^6$ *relations.* For the quadratic relations at this order, we have to consider the following products of generators,

$$\begin{aligned} {p_{(III)}}^i{}_j &= {A_{001}}_p{}^s {A_{002}}_q{}^i \epsilon^{pqr}\epsilon_{sri}\,, \\ {p_{(IV)}}^i{}_j &= {A_{002}}_p{}^s {A_{001}}_q{}^i \epsilon^{pqr}\epsilon_{sri}\,. \end{aligned} \tag{5.42}$$

The above generator products transform in the correct representation of SU(4) for this order, and satisfy the following quadratic relation,

$${P_{(II)}}^i{}_j = {p_{(III)}}^i{}_j + {p_{(IV)}}^i{}_j = 0\,. \tag{5.43}$$

The above is the correct relation for this order of the plethystic logarithm.

The next set of quadratic relations are of the order s^4t^6. In order to construct the quadratic relations at this order, we consider first the following SU(3) representation products,

$$\begin{aligned} [3,0]_x \times [1,1]_x &= [4,1]_x + [2,2]_x + [3,0]_x + [1,1]_x\,, \\ \mathrm{Sym}^2[1,1]_x &= [2,2]_x + [1,1]_x + [0,0]_x\,, \\ [1,1]_x \times [1,1]_x &= [2,2]_x + [3,0]_x + [0,3]_x + 2[1,1]_x + [0,0]_x\,. \end{aligned} \tag{5.44}$$

We make use of the above information to construct the quadratic relations at this order as follows:

- $-s^4t^6$ *relations.* For the quadratic relations at this order, we need to consider the following generator products,

$$\begin{aligned} u_{(I)} &= A_{002i}{}^{j} A_{002j}{}^{i} \,, \\ u_{(II)} &= (u_2 B)^2 \,, \end{aligned} \tag{5.45}$$

which transform as a singlet of SU(3). The above generator products form the following single unique quadratic relation,

$$U_{(I)} = u_{(I)} - \frac{2}{9} u_{(II)} = 0 \,, \tag{5.46}$$

which is the relation we are looking for at this order.

- $-2[1,1]_x s^4 t^6$ *relations.* The quadratic relations at this order can be constructed from the following generator products,

$$\begin{aligned} u_{(III)i}{}^{j} &= A_{002i}{}^{k} A_{002k}{}^{j} \,, \\ u_{(IV)i}{}^{j} &= A_{002i}{}^{j} A_{002k}{}^{k} \,, \\ u_{(V)i}{}^{j} &= u_3 A_{001i}{}^{j} B \,, \\ u_{(VI)i}{}^{j} &= u_2 A_{002i}{}^{j} B \,. \end{aligned} \tag{5.47}$$

All the above generator products transform in the adjoint of SU(3). They satisfy the following quadratic relations,

$$\begin{aligned} U_{(II)i}{}^{j} &= u_{(IV)i}{}^{j} - \frac{2}{3} u_{(VI)i}{}^{j} = 0 \,, \\ U_{(III)i}{}^{j} &= u_{(III)i}{}^{j} - \frac{1}{3} u_{(VI)i}{}^{j} - \frac{2}{9} u_{(V)i}{}^{j} = 0 \,. \end{aligned} \tag{5.48}$$

The above are precisely the two quadratic relations expected for this order.

- $-[2,2]_x s^4 t^6$ *relations.* For this order, the quadratic relation can be constructed by considering the following generator product and its index symmetrisations and anti-symmetrizations. The generator product to be considered is as follows,

$$u_{(VII)}{}^{ijklmn} = A_{001p}{}^{k} S_{012}{}^{lmn} \,, \tag{5.49}$$

which we anti-symmetrize in its indices $[kl]$ and $[mn]$,

$$u_{(VIII)}{}^{ijklmn} = u_{(VII)}{}^{ijklmn} - u_{(VII)}{}^{ijlkmn} - u_{(VII)}{}^{ijklnm} + u_{(VII)}{}^{ijlknm} \,. \tag{5.50}$$

We further symmetries the above in the pair of indices $[kl]$ and $[mn]$ as follows,

$$U_{(IV)}{}^{ijklmn} = u_{(VIII)}{}^{ijklmn} + u_{(VIII)}{}^{ijmnkl} = 0 \,, \tag{5.51}$$

which vanishes exactly. This is precisely the quadratic relations at this order.

- $-[3,0]_x s^4 t^6$ *relations.* The quadratic relation at this order can be obtained by considering the following product of generators and its symmetrisation of indices. The generator product to consider is,

$$u_{(IX)}{}^{ijk} = A_{001}{}_s{}^q S_{012}{}^{rjk} \epsilon^{sip} \epsilon_{qrp} , \tag{5.52}$$

where we symmetrize on the indices ijk to obtain,

$$U_{(V)}{}^{ijk} = u_{(IX)}{}^{ijk} + u_{(IX)}{}^{jki} + u_{(IX)}{}^{kij} = 0 . \tag{5.53}$$

The above vanishes exactly and corresponds precisely to the quadratic relation at this order.

The following set of quadratic relations involves the order $s^5 t^6$. In order to construct the relations at this order, we consider the following SU(3) representation products,

$$\begin{aligned} [3,0]_x \times [1,1]_x &= [4,1]_x + [2,2]_x + [3,0]_x + [1,1]_x , \\ \mathrm{Sym}^2[1,1]_x &= [2,2]_x + [1,1]_x + [0,0]_x , \\ [1,1]_x \times [1,1]_x &= [2,2]_x + [3,0]_x + [0,3]_x + 2[1,1]_x + [0,0]_x . \end{aligned} \tag{5.54}$$

The above representation products help us in finding the following quadratic relations between moduli space generators:

- $-[1,1]_x s^5 t^6$ *relations.* For the quadratic relation at this order, we consider first the following generator products,

$$\begin{aligned} v_{(I)}{}_i{}^j &= A_{002}{}_i{}^p S_{012}{}^{qrj} \epsilon_{pqr} , \\ v_{(II)}{}_i{}^j &= u_3 A_{001}{}_i{}^p A_{001}{}_p{}^j . \end{aligned} \tag{5.55}$$

The above products transform as desired in the adjoint of SU(3). Furthermore, the products satisfy the following quadratic relation,

$$V_{(I)}{}_i{}^j = v_{(I)}{}_i{}^j + \frac{1}{2} v_{(II)}{}_i{}^j = 0 , \tag{5.56}$$

which is precisely the relation we are looking for at this order.

- $-[2,2]_x s^5 t^6$ *relations.* For the quadratic relation at this order, we have to consider the following generator products with their index symmetrization and anti-symmetrizations. The first generator product to consider is the following,

$$v_{(III)}{}^{ijklmn} = A_{011}{}_p{}^k S_{012}{}^{lmn} \epsilon_{pij} , \tag{5.57}$$

which we anti-symmetrize in the indices $[kl]$ and $[mn]$ to get

$$v_{(IV)}{}^{ijklmn} = v_{(III)}{}^{ijklmn} - v_{(III)}{}^{ijlkmn} - v_{(III)}{}^{ijklnm} + v_{(III)}{}^{ijlknm} . \tag{5.58}$$

We further anti-symmetrize the above in the pairs of indices $[ij]$ and $[mn]$ as follows

$$v_{(V)}{}^{ijklmn} = v_{(IV)}{}^{ijklmn} - v_{(IV)}{}^{mnklij} . \tag{5.59}$$

The second generator product to consider is as follows

$$v_{(VI)}{}^{ijklmn} = u_3 A_{001p}{}^k A_{001r}{}^n \epsilon_{pij}\epsilon_{rlm}\,, \tag{5.60}$$

which we anti-symmetrize in the indices $[kl]$ and $[mn]$ to get

$$v_{(VII)}{}^{ijklmn} = v_{(VI)}{}^{ijklmn} - v_{(VI)}{}^{ijlkmn} - v_{(VI)}{}^{ijklnm} + v_{(VI)}{}^{ijlknm}\,. \tag{5.61}$$

We symmetrize the above product in the pairs of indices $[ij]$ and $[kl]$ to obtain

$$v_{(VIII)}{}^{ijklmn} = v_{(VII)}{}^{ijklmn} + v_{(VII)}{}^{klijmn}\,. \tag{5.62}$$

The above two generator products form the following quadratic relation,

$$V_{(II)}{}^{ijklmn} = v_{(V)}{}^{ijklmn} + \frac{1}{6} v_{(VIII)}{}^{ijmnkl} = 0\,, \tag{5.63}$$

which is precisely the relation we are looking for at this order.

- $-[3,0]_x s^5 t^6$ *relations.* The quadratic relation at this order can be found by considering the following generator products. The products to consider are

$$\begin{aligned} v_{(IX)}{}^{ijk} &= A_{002p}{}^p S_{012}{}^{ijk}\,,\\ v_{(X)}{}^{ijk} &= u_2 S_{012}{}^{ijk} B\,. \end{aligned} \tag{5.64}$$

The above two products satisfy the following quadratic relation,

$$V_{(III)}{}^{ijk} = v_{(IX)}{}^{ijk} - \frac{2}{3} v_{(X)}{}^{ijk} = 0\,, \tag{5.65}$$

which is precisely the relation we are looking for at this order.

The final quadratic relation is at order $s^6 t^6$. We consider the following SU(3) representation products for the construction of this relation,

$$\begin{aligned} \mathrm{Sym}^2[3,0]_x &= [6,0]_x + [2,2]_x\,,\\ [3,0]_x \times [1,1]_x &= [4,1]_x + [2,2]_x + [3,0]_x + [1,1]_x\,,\\ \mathrm{Sym}^2[1,1]_x &= [2,2]_x + [1,1]_x + [0,0]_x\,,\\ [1,1]_x \times [1,1]_x &= [2,2]_x + [3,0]_x + [0,3]_x + 2[1,1]_x + [0,0]_x\,. \end{aligned} \tag{5.66}$$

The above products help us in finding the correct generator product which leads to the following final relation between generators:

- $-[2,2]_x s^6 t^6$ *relations.* The final quadratic relations can be obtained by considering the following generator product,

$$z_{(I)}{}^{ijklmn} = S_{012}{}^{ijk} S_{012}{}^{lmn}\,, \tag{5.67}$$

which we anti-symmetrize in the indices $[kl]$ and $[mn]$ as follows

$$z_{(II)}{}^{ijklmn} = z_{(I)}{}^{ijklmn} - z_{(I)}{}^{ijlkmn} - z_{(I)}{}^{ijklnm} + z_{(I)}{}^{ijlknm}\,. \tag{5.68}$$

Another symmetrization of the pairs of indices $[kl]$ and $[mn]$ gives

$$Z^{ijklmn} = z_{(II)}{}^{ijklmn} + z_{(II)}{}^{ijmnkl} = 0\,, \tag{5.69}$$

which vanishes exactly and represents precisely the quadratic relation at this order.

Vortex moduli space. The vortex moduli space can be expressed as a $\mathbb{C}^*$ projection of the master space. The $\mathbb{C}^*$ projection gives as the vortex moduli space the following,

$$\widetilde{\mathcal{V}}_{3,3} = \widetilde{\mathcal{F}^\flat}_{3,3}/\{B_i \simeq \lambda^3 B\,, {A_{001}}_i{}^j \simeq \lambda^3 {A_{001}}_i{}^j\,, {A_{002}}_i{}^j \simeq \lambda^3 {A_{002}}_i{}^j\,, {S_{012}}^{ijk} \simeq \lambda^3 {S_{012}}^{ijk}\}\,. \tag{5.70}$$

The master space is expressed as follows,

$$\begin{aligned}\widetilde{\mathcal{F}^\flat}_{3,3} = \mathbb{C}[u_2, u_3, B, {A_{001}}_i{}^j, {A_{002}}_i{}^j, {S_{012}}^{ijk}]/\{ \\ & R_{(I)} = 0\,, {R_{(II)}}_i{}^j = 0\,, \\ & P_{(I)} = 0\,, {P_{(II)}}^i{}_j = 0\,, \\ & U_{(I)} = 0\,, {U_{(II)}}_i{}^j = 0\,, {U_{(III)}}_i{}^j = 0\,, {U_{(IV)}}^{ijklmn} = 0\,, {U_{(V)}}^{ijk} = 0\,, \\ & {V_{(I)}}_i{}^j = 0\,, {V_{(II)}}^{ijklmn} = 0\,, {V_{(I)}}^{ijk} = 0\,, \\ & Z^{ijklmn} = 0 \quad \}\,,\end{aligned} \tag{5.71}$$

where the quotient is taken by all quadratic relations of the master space generators.

5.4 3 U(4) vortices on $\mathbb{C}$

The Hilbert series for the 3 U(4) vortex master space is obtained from the following Molien integral

$$g(t, s, x; \widetilde{\mathcal{F}^\flat}_{3,4}) = \oint \mathrm{d}\mu_{\mathrm{SU}(3)} \mathrm{PE}\Big[[0,1]_w [1,0,0]_x t + [1,1]_w s\Big]\,, \tag{5.72}$$

where $[1,0,0]_x$ is the character of the fundamental representation of the global SU(4). The integral gives the following Hilbert series

$$\begin{aligned} g(t, s; \widetilde{\mathcal{F}^\flat}_{3,4}) = & \frac{1}{(1-s^2)(1-s^3)(1-t^3)^4(1-st^3)(1-s^3t^3)^4} \times \\ & (1 + 14st^3 + 20s^2t^3 + 16s^3t^3 - 16st^6 + 5s^2t^6 + 46s^3t^6 + 90s^4t^6 \\ & +60s^5t^6 + 10s^6t^6 + 4st^9 - 44s^2t^9 - 104s^3t^9 - 156s^4t^9 - 156s^5t^9 \\ & -60s^6t^9 - 24s^7t^9 + 21s^2t^{12} + 25s^3t^{12} + 23s^4t^{12} + 65s^5t^{12} - 73s^6t^{12} \\ & -195s^7t^{12} - 207s^8t^{12} - 81s^9t^{12} + s^{10}t^{12} + s^{11}t^{12} + 14s^3t^{15} \\ & +34s^4t^{15} + 22s^5t^{15} + 206s^6t^{15} + 438s^7t^{15} + 438s^8t^{15} + 206s^9t^{15} \\ & +22s^{10}t^{15} + 34s^{11}t^{15} + 14s^{12}t^{15} + s^4t^{18} + s^5t^{18} - 81s^6t^{18} - 207s^7t^{18} \\ & -195s^8t^{18} - 73s^9t^{18} + 65s^{10}t^{18} + 23s^{11}t^{18} + 25s^{12}t^{18} + 21s^{13}t^{18} \\ & -24s^8t^{21} - 60s^9t^{21} - 156s^{10}t^{21} - 156s^{11}t^{21} - 104s^{12}t^{21} - 44s^{13}t^{21} \\ & +4s^{14}t^{21} + 10s^9t^{24} + 60s^{10}t^{24} + 90s^{11}t^{24} + 46s^{12}t^{24} + 5s^{13}t^{24} \\ & -16s^{14}t^{24} + 16s^{12}t^{27} + 20s^{13}t^{27} + 14s^{14}t^{27} + s^{15}t^{30})\,, \end{aligned} \tag{5.73}$$

where for simplicity we set the global SU(4) fugacities to $x_1 = x_2 = x_3 = 1$. Accordingly, the master space of the vortex theory is a non-complete intersection Calabi-Yau space of dimension 13.

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As a character expansion, the Hilbert series is

$$g(t,s,x;\widetilde{\mathcal{F}^\flat}_{3,4}) = \frac{1}{(1-s^2)(1-s^3)}\times$$
$$\sum_{n_0=0}^{\infty}\sum_{n_1=0}^{\infty}\sum_{n_2=0}^{\infty}\sum_{n_3=0}^{\infty}\Bigg[[n_1+n_2+3n_3,n_1+n_2,n_0]_x s^{n_1+2n_2+3n_3}t^{3n_0+3n_1+3n_2+3n_3}$$
$$+[n_1+n_2,n_1+n_2+3n_3+3,n_0]_x s^{n_1+2n_2+3n_3+3}t^{3n_0+3n_1+3n_2+6n_3+6}\Bigg]. \tag{5.74}$$

The corresponding plethystic logarithm is

$$\begin{aligned}\mathrm{PL}\Big[g(t,s,x;\widetilde{\mathcal{F}^\flat}_{3,4})\Big] &= s^2+s^3+[0,0,1]_x t^3+[1,1,0]_x st^3+[1,1,0]_x s^2t^3+[3,0,0]_x s^3t^3\\ &\quad -([0,1,0]_x+[2,0,0]_x)st^6-([0,1,0]_x+2[2,0,0]_x+[0,0,2]_x\\ &\quad +[1,1,1]_x)s^2t^6-([0,1,0]_x+[0,0,2]_x+2[2,0,0]_x+2[1,1,1]_x\\ &\quad +[3,0,1]_x)s^3t^6-([2,0,0]_x+[0,0,2]_x+[1,1,1]_x+[2,2,0]_x\\ &\quad +[3,0,1]_x)s^4t^6+\dots\,.\end{aligned} \tag{5.75}$$

The generators of the vortex master space are encoded in the Hilbert series above. They are as follows,

$$\begin{aligned}
s^2 &\to u_2=\mathrm{Tr}(\phi^2)\\
s^3 &\to u_3=\mathrm{Tr}(\phi^3)\\
[0,0,1]_x t^3 &\to B_i=\epsilon^{\alpha_1\alpha_2\alpha_3}\epsilon_{k_1k_2k_3i}Q^{k_1}_{\alpha_1}Q^{k_2}_{\alpha_2}Q^{k_3}_{\alpha_3}\\
[1,1,0]_x st^3 &\to \begin{cases} {A_{001}}_{ij}{}^{k}=\epsilon^{\alpha_1\alpha_2\alpha_3}\epsilon_{k_1k_2ij}Q^{k_1}_{\alpha_1}Q^{k_2}_{\alpha_2}\phi^{\beta}_{\alpha_3}Q^{k}_{\beta}\\ {A_{001}}_{ik}{}^{k}=0\end{cases}\\
[1,1,0]_x s^2t^3 &\to \begin{cases} {A_{002}}_{ij}{}^{k}=\epsilon^{\alpha_1\alpha_2\alpha_3}\epsilon_{k_1k_2ij}Q^{k_1}_{\alpha_1}Q^{k_2}_{\alpha_2}\phi^{\beta_1}_{\alpha_3}\phi^{\beta_2}_{\beta_1}Q^{k}_{\beta_2}\\ 3{A_{002}}_{\ ik}{}^{k}=-u_2B_i\\ {A_{011}}_{ij}{}^{k}=\epsilon^{\alpha_1\alpha_2\alpha_3}\epsilon_{k_1k_2ij}Q^{k}_{\alpha_1}\phi^{\beta_1}_{\alpha_2}Q^{k_1}_{\beta_1}\phi^{\beta_2}_{\alpha_3}Q^{k_2}_{\beta_2}\\ 6{A_{011}}_{ik}{}^{k}=u_2B_i\\ \to A_{002ijk}=A_{011\ ijk}\\ \quad A_{002ikk}=A_{011\ ikk}-\frac{1}{6}u_2B_i\end{cases}\\
[3,0,0]_x s^3t^3 &\to \begin{cases} {A_{012}}^{ijk}=\epsilon^{\alpha_1\alpha_2\alpha_3}Q^{i}_{\alpha_1}\phi^{\beta_1}_{\alpha_2}Q^{j}_{\beta_1}\phi^{\beta_2}_{\alpha_3}\phi^{\beta_3}_{\beta_2}Q^{k}_{\beta_3}\\ {S_{012}}^{ijk}={A_{012}}^{ijk}+{A_{012}}^{jki}+{A_{012}}^{kij}\end{cases}\quad .
\end{aligned}$$

Quadratic relations. The terms in the plethystic logarithm corresponding to the quadratic relations between generators are as follows,

$$\begin{gathered}
-[0,1,0]_x st^6-[2,0,0]_x st^6\\
-[0,1,0]_x s^2t^6-2[2,0,0]_x s^2t^6-[0,0,2]_x s^2t^6-[1,1,1]_x s^2t^6\\
-[0,1,0]_x s^3t^6-[0,0,2]_x s^3t^6-2[2,0,0]_x s^3t^6-2[1,1,1]_x s^3t^6-[3,0,1]_x s^3t^6\\
-[2,0,0]_x s^4t^6-[0,0,2]_x s^4t^6-[1,1,1]_x s^4t^6-[2,2,0]_x s^4t^6-[3,0,1]_x s^4t^6\\
-[1,1,1]_x s^5t^6-[2,2,0]_x s^5t^6-[3,0,1]_x s^5t^6\\
-[2,2,0]_x s^6t^6\,.
\end{gathered} \tag{5.76}$$

From the discussion above regarding generators of the master space, we note that the quadratic relations can only be formed by the following generators

$$u_2\,, u_3\,, B_i\,, A_{001ij}{}^k\,, A_{002ij}{}^k\,, S_{012}{}^{ijk}\,. \tag{5.77}$$

Let us go through the quadratic relations corresponding to terms in (5.76) one by one.

The first relations to consider are at orders at st^6. For these relations, we consider the following product of SU(4) representations,

$$[1,1,0]_x \times [0,0,1]_x = [1,1,1]_x + [2,0,0]_x + [0,1,0]_x\,. \tag{5.78}$$

Using the above products of representations, we can easily construct the quadratic relations at this order. These are as follows:

- $-[0,1,0]_x st^6$ *relations.* The only non-trivial product of generators of the master space corresponding to the order $-[0,1,0]_x st^6$ is as follows,

  $$R_{(I)ij} = A_{001ij}{}^k\ B_k = 0\,, \tag{5.79}$$

  where it turns out that the products vanishes. Given that

  $$R_{(I)ij} = -R_{(I)ji}\,, \tag{5.80}$$

  this is the quadratic relation for $-[0,1,0]_x st^6$.

- $-[2,0,0]_x st^6$ *relations.* Another generator product which vanishes is

  $$A_{001ik}{}^k\ B_j = 0\,. \tag{5.81}$$

  This product can be symmetrised as follows

  $$R_{(II)ij} = A_{001ik}{}^k\ B_j + A_{001jk}{}^k\ B_i = 0\,, \tag{5.82}$$

  such that

  $$R_{(II)ij} = R_{(II)ji}\,, \tag{5.83}$$

  and it corresponds to quadratic relations of the order $-[2,0,0]_x st^6$.

The second set of quadratic relations are at orders of s^2t^6. For these quadratic relations, we consider the following products of SU(4) representations,

$$\begin{aligned}
\mathrm{Sym}^2[1,1,0]_x &= [2,2,0]_x + [1,1,1]_x + [2,0,0]_x + [0,0,2]_x\,,\\
\mathrm{Sym}^2[0,0,1]_x &= [0,0,2]_x\,,\\
[1,1,0]_x \times [0,0,1]_x &= [1,1,1]_x + [2,0,0]_x + [0,1,0]_x\,.
\end{aligned} \tag{5.84}$$

The above product decompositions of SU(4) representations help us to construct the quadratic relations at this order as follows:

- $-[0,1,0]_x s^2 t^6$ *relations.* The only product of master space generators that can correspond to order $-[0,1,0]_x s^2 t^6$ is the following,

$$O_{(I)_{ij}} = A_{002_{ij}}{}^{k}\ B_k = 0\,, \tag{5.85}$$

which vanishes exactly. The above satisfies

$$O_{(I)_{ij}} = -O_{(I)_{ji}}\,, \tag{5.86}$$

such that this is the quadratic relation at order $-[0,1,0]_x s^2 t^6$.

- $-2[2,0,0]_x s^2 t^6$ *relations.* The following generator products relate to the o rder $-[2,0,0]_x s^2 t^6$,

$$\begin{aligned} o_{(I)_{ij}} &= A_{002_{ik}}{}^{k} B_j + A_{002_{jk}}{}^{k} B_i\,, \\ o_{(II)_{ij}} &= u_2 B_i B_j\,, \\ o_{(III)_{ij}} &= A_{001_{ik}}{}^{m} A_{001_{jm}}{}^{k}\,, \end{aligned} \tag{5.87}$$

where for $o_{(I)_{ij}}$ we have symmetrized the product

$$A_{002_{ik}}{}^{k} B_j\,. \tag{5.88}$$

We have

$$o_{(I)_{ij}} = o_{(I)_{ji}}\,, o_{(II)_{ij}} = o_{(II)_{ji}}\,, o_{(III)_{ij}} = o_{(III)_{ji}}\,. \tag{5.89}$$

From the above products, we can identify the following independent quadratic relations,

$$\begin{aligned} O_{(II)_{ij}} &= o_{(I)_{ij}} + 6 o_{(III)_{ij}} = 0 \\ O_{(III)_{ij}} &= o_{(II)_{ij}} - 9 o_{(III)_{ij}} = 0\,, \end{aligned} \tag{5.90}$$

which correspond to the order $-2[2,0,0]_x s^2 t^6$ of the plethystic logarithm.

- $-[0,0,2]_x s^2 t^6$ *relations.* The following generator products correspond to the order $-[0,0,2]_s^2 t^6$

$$\begin{aligned} o_{(IV)_{pqijlm}} &= \epsilon_{pqrs} A_{001_{ij}}{}^{r} A_{001_{lm}}{}^{s} \\ o_{(V)_{pqijlm}} &= \epsilon_{pqrm} A_{002_{ij}}{}^{r} B_l\,. \end{aligned} \tag{5.91}$$

The above products satisfy the following the quadratic relation,

$$O_{(IV)_{pqijlm}} = 3 o_{(IV)_{pqijlm}} - o_{(V)_{pqijlm}} = 0\,, \tag{5.92}$$

which corresponds to the order $-[0,0,2]_x s^2 t^6$.

- $-[1,1,1]_x s^2 t^6$ *relations.* The following generator product is the only one which corresponds to the order $-[1,1,1]_x s^2 t^6$ of the plethystic logarithm of the master space Hilbert series,

$$O_{(V)ij}{}^{k}{}_{l} = A_{001ij}{}^{k} A_{001lm}{}^{m} = 0\,. \tag{5.93}$$

The third set of quadratic relation of generators at orders of s^3t^6 involve the following representation products,

$$\begin{aligned}
\mathrm{Sym}^2[0,0,1]_x &= [0,0,2]_x\,,\\
[1,1,0]_x \times [1,1,0]_x &= [2,2,0]_x + [3,0,1]_x + [0,3,0]_x + [1,1,1]_x + [2,0,0]_x\\
&\quad +[0,0,2]_x + [0,1,0]_x\,,\\
[1,1,0]_x \times [0,0,1]_x &= [1,1,1]_x + [2,0,0]_x + [0,1,0]_x\,,\\
[3,0,0]_x \times [0,0,1]_x &= [3,0,1]_x + [2,0,0]_x\,.
\end{aligned} \tag{5.94}$$

The above SU(4) representation products are used to identify the quadratic relations at this order as follows:

- $-[0,1,0]_x s^3 t^6$ *relations.* At this order, there is only the following generator product,

$$P_{(I)ij} = A_{001ij}{}^{k}\, B_k = 0\,, \tag{5.95}$$

which exactly vanishes. The relation satisfies,

$$P_{(I)ij} = -P_{(I)ji}\,, \tag{5.96}$$

such that the quadratic relation corresponds to the order $-[0,1,0]_x s^3 t^6$.

- $-[0,0,2]_x s^3 t^6$ *relations.* For the quadratic relations at this order, we have to consider the following generator products,

$$\begin{aligned}
p_{(I)ij} &= A_{001ip}{}^{q} A_{002jq}{}^{p}\,,\\
p_{(II)ij} &= u_3 B_i B_j\,.
\end{aligned} \tag{5.97}$$

The above products transform in $[0,0,2]_x$ and satisfy the following quadratic relation

$$P_{(II)ij} = p_{(I)ij} - \frac{1}{9} p_{(II)ij} = 0\,, \tag{5.98}$$

which is precisely the relation we are looking for at this order.

- $-2[2,0,0]_x s^3 t^6$ *relations.* One of the generator products which corresponds to the order $-[2,0,0]_x s^2 t^6$ is as follows,

$$P_{(III)}{}^{ij} = S_{012}{}^{ijk} B_k = 0\,, \tag{5.99}$$

which vanishes exactly. The above satisfies

$$P_{(III)}{}^{ij} = P_{(III)}{}^{ji}\,, \tag{5.100}$$

and is the first quadratic relation corresponding to $-[2,0,0]_x s^2 t^6$. Another set of generator products which correspond to this order is as follows,

$$\begin{aligned} p_{(III)}{}_{ij} &= A_{002}{}_{ik}{}^{m} A_{001}{}_{im}{}^{k} \\ p_{(IV)}{}_{ij} &= u_3 B_i B_j \,. \end{aligned} \tag{5.101}$$

These products satisfy the following quadratic relation,

$$P_{(IV)}{}_{ij} = 9 p_{(III)}{}_{ij} - p_{(IV)}{}_{ij} = 0\,, \tag{5.102}$$

which is the second relation corresponding to the order $-[2,0,0]_x s^2 t^6$.

- $-2[1,1,1]_x s^3 t^6$ *relations.* The following generator products transform in $-[1,1,1]_x s^3 t^6$,

$$\begin{aligned} p_{(V)}{}_{jm}{}^{l}{}_{i} &= u_2 A_{001}{}_{jm}{}^{l} B_i \,, \\ p_{(VI)}{}_{jm}{}^{l}{}_{i} &= A_{002}{}_{jm}{}^{k} A_{001}{}_{ki}{}^{l} \,, \\ p_{(VII)}{}_{jm}{}^{l}{}_{i} &= A_{001}{}_{jm}{}^{k} A_{002}{}_{ki}{}^{l} \,. \end{aligned} \tag{5.103}$$

They satisfy the following quadratic relations,

$$\begin{aligned} P_{(V)}{}_{jm}{}^{l}{}_{i} &= p_{(V)}{}_{jm}{}^{l}{}_{i} - 6 p_{(VI)}{}_{jm}{}^{l}{}_{i} \,, \\ P_{(VI)}{}_{jm}{}^{l}{}_{i} &= p_{(V)}{}_{jm}{}^{l}{}_{i} - 6 p_{(VII)}{}_{jm}{}^{l}{}_{i} \,, \end{aligned} \tag{5.104}$$

which satisfy

$$P_{(V)}{}_{jm}{}^{l}{}_{i} = -P_{(V)}{}_{jm}{}^{l}{}_{i} \,, P_{(VI)}{}_{jm}{}^{l}{}_{i} = -P_{(VI)}{}_{jm}{}^{l}{}_{i} \,, \tag{5.105}$$

and hence correspond to the two quadratic relations at order $-[1,1,1]_x s^3 t^6$.

- $-[3,0,1]_x s^3 t^6$ *relations.* For the quadratic relation at this order, we consider the following generator products. The first one to consider is,

$$p_{(VIII)}{}^{ijk}{}_{m} = A_{001}{}_{pq}{}^{i} A_{002}{}_{rn}{}^{j} \epsilon^{pqrk} \,, \tag{5.106}$$

which we symmetrize as follows,

$$p_{(IX)}{}^{ijk}{}_{m} = p_{(VIII)}{}^{ijk}{}_{m} + p_{(VIII)}{}^{jki}{}_{m} + p_{(VIII)}{}^{kij}{}_{m} \,. \tag{5.107}$$

The second product to consider is

$$p_{(X)}{}^{ijk}{}_{m} = S_{012}{}^{ijk} B_m \,. \tag{5.108}$$

The above generator products satisfy the following quadratic relation

$$P_{(VII)}{}^{ijk}{}_{m} = p_{(IX)}{}^{ijk}{}_{m} - p_{(X)}{}^{ijk}{}_{m} = 0\,, \tag{5.109}$$

which is the relation we are looking for at this order.

The next set of quadratic relations at orders of s^4t^6 can be identified by considering the following products of representations,

$$\begin{aligned}
\mathrm{Sym}^2[1,1,0]_x &= [2,2,0]_x + [1,1,1]_x + [2,0,0]_x + [0,0,2]_x\,,\\
\mathrm{Sym}^2[0,0,1]_x &= [0,0,2]_x\,,\\
[3,0,0]_x \times [1,1,0]_x &= [3,1,1]_x + [2,0,2]_x + [2,1,0]_x + [1,0,1]_x\,,\\
[1,1,0]_x \times [0,0,1]_x &= [1,1,1]_x + [2,0,0]_x + [0,1,0]_x\,.
\end{aligned} \tag{5.110}$$

We make use of the above products to construct the quadratic relations of this order as follows:

- $-[2,0,0]_x s^4t^6$ *relations.* For order $-[2,0,0]_x s^4t^6$, the following generator product applies,

$$U_{(I)}{}^{ij} = \epsilon^{pqrs} A_{002pq}{}^{i} A_{002rs}{}^{j} = 0\,, \tag{5.111}$$

which vanishes exactly and satisfies,

$$U_{(I)}{}^{ij} = U_{(I)}{}^{ji}\,. \tag{5.112}$$

This is exactly the quadratic relation corresponding to the order $-[2,0,0]_x s^4t^6$.

- $-[0,0,2]_x s^4t^6$ *relations.* For the quadratic relation at this order, we need to first consider the following products of generators,

$$\begin{aligned}
u_{(I)}{}^{ij} &= A_{002ip}{}^{q} A_{002jq}{}^{p}\,,\\
u_{(II)}{}^{ij} &= (u_2)^2 B_i B_j\,.
\end{aligned} \tag{5.113}$$

The above products satisfy the following quadratic relation of this order,

$$U_{(II)}{}^{ij} = u_{(I)}{}^{ij} - \frac{1}{18} u_{(II)}{}^{ij} = 0\,, \tag{5.114}$$

which is the one we are looking for.

- $-[1,1,1]_x s^4t^6$ *relations.* The following generator products correspond to this order of the plethystic logarithm,

$$\begin{aligned}
u_{(III)ij}{}^{k}{}_{l} &= A_{002ij}{}^{k} A_{002lm}{}^{m}\,,\\
u_{(IV)ij}{}^{k}{}_{l} &= u_3 A_{001ij}{}^{k} B_l\,,\\
u_{(V)ij}{}^{k}{}_{l} &= u_2 A_{002ij}{}^{k} B_l\,.
\end{aligned} \tag{5.115}$$

The above products satisfy the following quadratic relation,8

$$U_{(III)ij}{}^{k}{}_{l} = u_{(III)ij}{}^{k}{}_{l} + 13 u_{(IV)ij}{}^{k}{}_{l} - u_{(V)ij}{}^{k}{}_{l} = 0\,, \tag{5.116}$$

which corresponds to the order $-[1,1,1]_x s^4t^6$.

- $-[2,2,0]_x s^4 t^6$ *relations.* For this order, the quadratic relations can be constructed by looking at generator products and their index symmetrizations and anti-symmetrizations. We consider

$$u_{(VI)}{}^{ijklmn} = \epsilon^{ijpq} A_{001}{}_{pq}{}^{k} S_{012}{}^{lmn} \ . \tag{5.117}$$

which we anti-symmetrize in the indices $[kl]$ and $[mn]$ as follows,

$$u_{(VII)}{}^{ijklmn} = u_{(VI)}{}^{ijklmn} - u_{(VI)}{}^{ijlkmn} - u_{(VI)}{}^{ijklnm} + u_{(VI)}{}^{ijlknm} \ . \tag{5.118}$$

We follow with a further symmetrization of the pairs of indices $[kl]$ and $[mn]$

$$U_{(IV)}{}^{ijklmn} = u_{(VII)}{}^{ijklmn} + u_{(VII)}{}^{ijmnkl} = 0 \, . \tag{5.119}$$

The above vanishes exactly and forms the required quadratic relation at this order.

- $-[3,0,1]_x s^4 t^6$ *relations.* For the quadratic relations at this order, we consider first the following product of generators of the vortex master space,

$$u_{(VIII)}{}^{ijk}{}_{m} = A_{001}{}_{wv}{}^{q} S_{012}{}^{rjk} \epsilon^{wvip} \epsilon_{pqrm} \, . \tag{5.120}$$

We symmetrize the above product as follows,

$$U_{(V)}{}^{ijk}{}_{m} = u_{(VIII)}{}^{ijk}{}_{m} + u_{(VIII)}{}^{jki}{}_{m} + u_{(VIII)}{}^{kij}{}_{m} = 0 \, , \tag{5.121}$$

where we see that the symmetrization vanishes non-trivially. This is precisely the relation related to the order $-[3,0,1]_x s^4 t^6$ of the plethystic logarithm of the Hilbert series.

Let us move on to the next set of quadratic relations at orders of $s^5 t^6$. For these relations, we consider the following products of SU(4) representations,

$$\begin{aligned}
\mathrm{Sym}^2[1,1,0]_x &= [2,2,0]_x + [1,1,1]_x + [2,0,0]_x + [0,0,2]_x \, , \\
\mathrm{Sym}^2[0,0,1]_x &= [0,0,2]_x \, , \\
[3,0,0]_x \times [1,1,0]_x &= [3,1,1]_x + [2,0,2]_x + [2,1,0]_x + [1,0,1]_x \, , \\
[1,1,0]_x \times [0,0,1]_x &= [1,1,1]_x + [2,0,0]_x + [0,1,0]_x \, , \\
[1,1,0]_x \times [1,1,0]_x &= [2,2,0]_x + [3,0,1]_x + [0,3,0]_x + 2[1,1,1]_x \\
&\quad + [2,0,0]_x + [0,0,2]_x \, .
\end{aligned} \tag{5.122}$$

We use the above products of representations in order to construct the quadratic relations of generators at this order as follows:

- $-[1,1,1]_x s^5 t^6$ *relations.* For the quadratic relations at this order, we need to consider the following products of generators,

$$\begin{aligned}
v_{(I)}{}_{ij}{}^{k}{}_{m} &= A_{002}{}_{ij}{}^{p} S_{012}{}^{qrk} \epsilon_{pqrm} \, , \\
v_{(II)}{}_{ij}{}^{k}{}_{m} &= u_3 A_{001}{}_{ij}{}^{p} A_{001}{}_{pm}{}^{k} \, .
\end{aligned} \tag{5.123}$$

These transform in the correct SU(4) representation of this order. We find that they satisfy the following quadratic relations,

$$V_{(I)}{}_{ij}{}^{k}{}_{m} = v_{(I)}{}_{ij}{}^{k}{}_{m} + \frac{1}{2} v_{(II)}{}_{ij}{}^{k}{}_{m} \,. \tag{5.124}$$

This is precisely the relation of this order.

- $-[2,2,0]_x s^5 t^6$ *relations.* The quadratic relation at this order is constructed by considering the following products of master space generators, with their index symmetrizations and anti-symmetrizations. The first generator product to consider is,

$$v_{(III)}{}^{ijklmn} = A_{011}{}_{pq}{}^{k} S_{012}{}^{lmn} \epsilon^{pqij} \,, \tag{5.125}$$

which we anti-symmetrize on the indices $[kl]$ and $[mn]$ as follows,

$$v_{(IV)}{}^{ijklmn} = v_{(III)}{}^{ijklmn} - v_{(III)}{}^{ijlkmn} - v_{(III)}{}^{ijklnm} + v_{(III)}{}^{ijlknm} \,. \tag{5.126}$$

A further anti-symmetrization on the pair of indices $[ij]$ and $[mn]$ gives

$$v_{(V)}{}^{ijklmn} = v_{(IV)}{}^{ijklmn} - v_{(IV)}{}^{mnklij} \,. \tag{5.127}$$

The second generator product to consider is as follows,

$$v_{(VI)}{}^{ijklmn} = u_3 A_{001}{}_{pq}{}^{k} A_{001}{}_{rs}{}^{n} \epsilon^{pqij} \epsilon^{rslm} \,. \tag{5.128}$$

We anti-symmetrize the above product in the indices $[kl]$ and $[mn]$,

$$v_{(VII)}{}^{ijklmn} = v_{(VI)}{}^{ijklmn} - v_{(VI)}{}^{ijlkmn} - v_{(VI)}{}^{ijklnm} + v_{(VI)}{}^{ijlknm} \,, \tag{5.129}$$

and further symmetrize the product in the pairs of indices $[ij]$ and $[kl]$ as follows

$$v_{(VIII)}{}^{ijklmn} = v_{(VII)}{}^{ijklmn} + v_{(VII)}{}^{klijmn} \,. \tag{5.130}$$

Using the above generator products, we identify the following quadratic relation,

$$V_{(II)}{}^{ijklmn} = v_{(V)}{}^{ijklmn} + \frac{1}{4} v_{(VIII)}{}^{ijmnkl} \,, \tag{5.131}$$

which is the relation we are looking for at this order.

- $-[3,0,1]_x s^5 t^6$ *relations.* For the quadratic relations at this order, we have to consider the following generator products with their symmetrizations of indices. The first product to consider is as follows,

$$v_{(IX)}{}^{ijk}{}_{m} = A_{002}{}_{vw}{}^{q} S_{012}{}^{rjk} \epsilon^{vwip} \epsilon_{pqrm} \,, \tag{5.132}$$

which we symmetrize in the indices ijk such that

$$v_{(X)}{}^{ijk}{}_{m} = v_{(IX)}{}^{ijk}{}_{m} + v_{(IX)}{}^{jki}{}_{m} + v_{(IX)}{}^{kij}{}_{m} \,. \tag{5.133}$$

The second generator product to consider is

$$v_{(XI)}{}^{ijk}{}_m = u_2 S_{012}{}^{ijk} B_m \,. \tag{5.134}$$

With the first one, the above product satisfies the following quadratic relation,

$$V_{(III)}{}^{ijk}{}_m = v_{(X)}{}^{ijk}{}_m - \frac{4}{3} v_{(XI)}{}^{ijk}{}_m = 0 \,, \tag{5.135}$$

which is precisely the relation we are looking for at this order of the plethystic logarithm.

For the final quadratic relation at the order of s^6t^6, we construct candidates for which the following products of SU(4) representations are useful,

$$\begin{aligned}
\mathrm{Sym}^2[3,0,0]_x &= [6,0,0]_x + [2,2,0]_x \,, \\
\mathrm{Sym}^2[1,1,0]_x &= [2,2,0]_x + [1,1,1]_x + [2,0,0]_x + [0,0,2]_x \,, \\
[3,0,0]_x \times [1,1,0]_x &= [4,1,0]_x + [2,2,0]_x + [3,0,1]_x + [1,1,1]_x \,, \\
[1,1,0]_x \times [1,1,0]_x &= [2,2,0]_x + [3,0,1]_x + [0,3,0]_x + [1,1,1]_x \\
&\quad + [2,0,0]_x + [0,0,2]_x + [0,1,0]_x \,.
\end{aligned} \tag{5.136}$$

The above representation products can be used to identify the correct generator products and to construct the following quadratic relation at this order:

- $-[2,2,0]_x s^6 t^6$ *relations.* The quadratic relation at this order can be obtained by considering the following generator product,

$$z_{(I)}{}^{ijklmn} = S_{012}{}^{ijk} S_{012}{}^{lmn} \,. \tag{5.137}$$

We first anti-symmetrize the indices $[kl]$ and $[mn]$ as follows,

$$z_{(II)}{}^{ijklmn} = z_{(I)}{}^{ijklmn} - z_{(I)}{}^{ijlkmn} - z_{(I)}{}^{ijklnm} + z_{(I)}{}^{ijlknm} \,, \tag{5.138}$$

and then symmetrize the pairs of indices $[kl]$ and $[mn]$ to obtain

$$Z^{ijklmn} = z_{(II)}{}^{ijklmn} + z_{(II)}{}^{ijmnkl} = 0 \,, \tag{5.139}$$

which vanishes non-trivially. As a result, we identify the above as the desired quadratic relation at this order of the plethystic logarithm.

Vortex moduli space. The vortex moduli space for 3 U(4) vortices can be expressed as a partial $\mathbb{C}^*$ projection of the vortex master space. The projection is given as follows,

$$\begin{aligned}
\mathcal{V}_{3,4} = \mathcal{F}^\flat_{3,4} / \{ & B_i \simeq \lambda^3 B_i, A_{001ij}{}^k \simeq \lambda^3 A_{001ij}{}^k, \\
& A_{002ij}{}^k \simeq \lambda^3 A_{002ij}{}^k, S_{012}{}^{ijk} \simeq \lambda^3 S_{012}{}^{ijk} \} \,,
\end{aligned} \tag{5.140}$$

where λ is the $\mathbb{C}^*$ parameter. The dimension of the partially projected space representing the vortex moduli space is as expected 12. The master space is expressed as the following quotient,

$$\begin{aligned}
\mathcal{F}^\flat_{3,4} = \mathbb{C}[u_2, u_3, B_i, {A_{001}}_{ij}{}^k, {A_{002}}_{ij}{}^k, {S_{012}}^{ijk}]/\{ \\
& R_{(I)ij} = 0\,, R_{(II)ij} = 0\,, \\
& O_{(I)ij} = 0\,, O_{(II)ij} = 0\,, O_{(III)ij} = 0\,, O_{(IV)pqijlm} = 0\,, O_{(V)ij}{}^k{}_l = 0\,, \\
& P_{(I)ij} = 0\,, P_{(II)ij} = 0\,, P_{(III)}{}^{ij} = 0\,, P_{(IV)ij} = 0\,, P_{(V)jm}{}^l{}_i = 0\,, \\
& P_{(VI)jm}{}^l{}_i = 0\,, P_{(VII)}{}^{ijk}{}_m = 0\,, \\
& U_{(I)}{}^{ij} = 0\,, U_{(II)}{}^{ij} = 0\,, U_{(III)ij}{}^k{}_l = 0\,, U_{(IV)}{}^{ijklmn} = 0\,, U_{(V)}{}^{ijk}{}_m = 0\,, \\
& V_{(I)ij}{}^k{}_m = 0\,, V_{(II)}{}^{ijklmn} = 0\,, V_{(III)}{}^{ijk}{}_m = 0\,, \\
& Z^{ijklmn} = 0 \quad \}\,.
\end{aligned} \tag{5.141}$$

5.5 3 U(5) vortices on $\mathbb{C}$

The Hilbert series for the 3 U(5) reduced vortex master space is obtained from the following Molien integral,

$$g(t,s,x;\widetilde{\mathcal{F}^\flat}_{3,5}) = \oint \mathrm{d}\mu_{\mathrm{SU}(3)} \mathrm{PE}\Big[[0,1]_w[1,0,0,0]_x t + [1,1]_w s\Big]\,, \tag{5.142}$$

where $[1,0,0,0]_x$ is the character of the fundamental representation of the global SU(5). The integral leads to the following character expansion of the Hilbert series

$$\begin{aligned}
g(t,s,x;\widetilde{\mathcal{F}^\flat}_{3,5}) &= \frac{1}{(1-s^2)(1-s^3)} \times \\
\sum_{n_0=0}^{\infty}\sum_{n_1=0}^{\infty}\sum_{n_2=0}^{\infty}\sum_{n_3=0}^{\infty} & \Big[[n_1+n_2+3n_3, n_1+n_2, n_0, 0]_x s^{n_1+2n_2+3n_3} t^{3n_0+3n_1+3n_2+3n_3} \\
& + [n_1+n_2, n_1+n_2+3n_3+3, n_0, 0]_x s^{n_1+2n_2+3n_3+3} t^{3n_0+3n_1+3n_2+6n_3+6}\Big]\,,
\end{aligned} \tag{5.143}$$

where $[m_1, m_2, m_3, m_4]_x$ is a character of a SU(5) irreducible representation.

The plethystic logarithm of the Hilbert series is

$$\begin{aligned}
\mathrm{PL}\Big[g(t,s,x;\widetilde{\mathcal{F}^\flat}_{3,5})\Big] &= s^2 + s^3 + [0,0,1,0]_x t^3 + [1,1,0,0]_x s t^3 + [1,1,0,0]_x s^2 t^3 \\
& +[3,0,0,0]_x s^3 t^3 - [1,0,0,0]_x t^6 - ([2,0,0,1]_x + [0,1,0,1]_x \\
& +[1,0,0,0]_x) s t^6 - ([1,1,1,0]_x + [0,0,2,0]_x + 2[2,0,0,1]_x \\
& +[0,1,0,1]_x + [1,0,0,0]_x) s^2 t^6 - ([0,3,0,0]_x + [3,0,1,0]_x \\
& +2[1,1,1,0]_x + [0,0,2,0]_x + 2[2,0,0,1]_x + [0,1,0,1]_x) s^3 t^6 \\
& -([2,2,0,0]_x + [3,0,1,0]_x + [0,0,2,0]_x + [2,0,0,1]_x \\
& +2[1,1,1,0]_x) s^4 t^6 + \ldots \\
& -([1,1,1,0]_x + [2,2,0,0]_x + [3,0,1,0]_x) s^5 t^6 + \ldots \\
& -[2,2,0,0]_x s^6 t^6 + \ldots\,.
\end{aligned} \tag{5.144}$$

The generators of the vortex moduli space are indicated by the above plethystic logarithm. They are as follows,

$$
\begin{aligned}
s^2 &\rightarrow u_2 = \mathrm{Tr}(\phi^2)\\
s^3 &\rightarrow u_3 = \mathrm{Tr}(\phi^3)\\
[0,0,1,0]_x t^3 &\rightarrow B_{ij} = \epsilon^{\alpha_1\alpha_2\alpha_3}\epsilon_{k_1k_2k_3ij}Q^{k_1}_{\alpha_1}Q^{k_2}_{\alpha_2}Q^{k_3}_{\alpha_3}\\
[1,1,0,0]_x st^3 &\rightarrow \begin{cases} A_{001}{}^{ijk} = \epsilon^{\alpha_1\alpha_2\alpha_3}Q^i_{\alpha_1}Q^j_{\alpha_2}\phi^{\beta}_{\alpha_3}Q^k_{\beta}\\ \epsilon_{ijkmn}A_{001}{}^{ijk} = 0\end{cases}\\
[1,1,0,0]_x s^2t^3 &\rightarrow \begin{cases} A_{002}{}^{ijk} = \epsilon^{\alpha_1\alpha_2\alpha_3}Q^i_{\alpha_1}Q^j_{\alpha_2}\phi^{\beta_1}_{\alpha_3}\phi^{\beta_2}_{\beta_1}Q^k_{\beta_2}\\ \epsilon_{ijkmn}A_{002}{}^{ijk} = \frac{1}{3}u_2B_{mn}\\ A_{011}{}^{ijk} = \epsilon^{\alpha_1\alpha_2\alpha_3}Q^k_{\alpha_1}\phi^{\beta_1}_{\alpha_2}Q^i_{\beta_1}\phi^{\beta_2}_{\alpha_3}Q^j_{\beta_2}\\ \epsilon_{ijkmn}A_{011}{}^{ijk} = -\frac{1}{6}u_2B_{mn}\\ \rightarrow A_{002}{}^{ijk} = A_{011}{}^{ijk} + \frac{1}{24}u_2\epsilon^{ijkmn}B_{mn}\end{cases}\\
[3,0,0,0]_x s^3t^3 &\rightarrow \begin{cases} A_{012}{}^{ijk} = \epsilon^{\alpha_1\alpha_2\alpha_3}Q^i_{\alpha_1}\phi^{\beta_1}_{\alpha_2}Q^j_{\beta_1}\phi^{\beta_2}_{\alpha_3}\phi^{\beta_3}_{\beta_2}Q^k_{\beta_3}\\ S_{012}{}^{ijk} = A_{012}{}^{ijk} + A_{012}{}^{jki} + A_{012}{}^{kij}\end{cases} .
\end{aligned}
\tag{5.145}
$$

Quadratic relations. The terms of the plethystic logarithm corresponding to quadratic relations between master space generators are as follows,

$$
\begin{gathered}
-[1,0,0,0]_x t^6\\
-[0,1,0,1]_x st^6 - [2,0,0,1]_x st^6 - [1,0,0,0]_x st^6\\
-[0,1,0,1]_x s^2t^6 - 2[2,0,0,1]_x s^2t^6 - [0,0,2,0]_x s^2t^6 - [1,1,1,0]_x s^2t^6 - [1,0,0,0]_x s^2t^6\\
-[0,1,0,1]_x s^3t^6 - 2[2,0,0,1]_x s^3t^6 - [0,0,2,0]_x s^3t^6 - 2[1,1,1,0]_x s^3t^6 - [3,0,1,0]_x s^3t^6\\
-[2,0,0,1]_x s^4t^6 - [0,0,2,0]_x s^4t^6 - 2[1,1,1,0]_x s^4t^6 - [2,2,0,0]_x s^4t^6 - [3,0,1,0]_x s^4t^6\\
-[1,1,1,0]_x s^5t^6 - [2,2,0,0]_x s^5t^6 - [3,0,1,0]_x s^5t^6\\
-[2,2,0,0]_x s^6t^6 .
\end{gathered}
\tag{5.146}
$$

From (5.145), we remind ourselves about the generators of the master space which form the quadratic relations corresponding to the negative terms of the plethystic logarithm in (5.146) above,

$$u_2\,, u_3\,, B_{ij}\,, A_{001}{}^{ijk}\,, A_{002}{}^{ijk}\,, S_{012}{}^{ijk} . \tag{5.147}$$

Let us go through the quadratic relations as follows, by first considering the relations associated to the order t^6. At this order, we have to consider the following symmetric product of SU(5) representations,

$$\mathrm{Sym}^2[0,0,1,0]_x = [0,0,2,0]_x + [1,0,0,0]_x\,, \tag{5.148}$$

which leads to the following relations:

- $-[1,0,0,0]_x t^6$ *relations.* The following generator product corresponds to this order,

$$H^i = \epsilon^{ijklm}B_{jk}B_{lm} = 0\,. \tag{5.149}$$

The product vanishes exactly and corresponds to the quadratic relation at this order.

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The next order of relations to consider is st^6. For this order, we consider the following representation product,

$$[1,1,0,0]_x \times [0,0,1,0]_x = [1,1,1,0]_x + [2,0,0,1]_x + [0,1,0,1]_x + [1,0,0,0]_x \,. \quad (5.150)$$

From the above product, we construct the following quadratic relations of generators for orders of st^6:

- $-[0,1,0,1]_x st^6$ *relations.* The following product of generators transforms in $-[0,1,0,1]_x st^6$,

$$R_{(I)}{}^{ij}{}_k = A_{001}{}^{ijm} B_{mk} = 0 \,, \quad (5.151)$$

  which exactly vanishes. This is precisely the quadratic relations at this order of the plethystic logarithm.

- $-[2,0,0,1]_x st^6$ *relations.* For the quadratic relation at this order, we consider the following generator product,

$$R_{(II)}{}^{ij}{}_k = A_{001}{}^{imj} B_{mk} = 0 \,, \quad (5.152)$$

  which exactly vanishes and accordingly corresponds to the quadratic relation at order $-[2,0,0,1]_x st^6$.

- $-[1,0,0,0]_x st^6$ *relations.* At this order, we have the following generator product,

$$R_{(III)}{}^{i} = A_{001}{}^{imn} B_{mn} = 0 \,, \quad (5.153)$$

  which vanishes exactly. This is the quadratic relation which transforms in $-[1,0,0,0]_x st^6$.

The next order to consider is s^2t^6. For this order, we have to look at the following SU(5) representation products,

$$\begin{aligned} \mathrm{Sym}^2[1,1,0,0]_x &= [2,2,0,0]_x + [1,1,1,0]_x + [2,0,0,1]_x + [0,0,2,0]_x \,, \\ [1,1,0,0]_x \times [0,0,1,0]_x &= [1,1,0,0]_x + [2,0,0,1]_x + [0,1,0,1]_x + [1,0,0,0]_x \,, \\ \mathrm{Sym}^2[0,0,1,0]_x &= [0,0,2,0]_x + [1,0,0,0]_x \,. \end{aligned} \quad (5.154)$$

The above products lead to the following quadratic relations:

- $-[0,1,0,1]_x s^2t^6$ *relations.* The following generator product corresponds to this order,

$$O_{(I)}{}^{ij}{}_k = A_{002}{}^{ijm} B_{mk} \,, \quad (5.155)$$

  and vanishes exactly. This is the quadratic relation corresponding to $-[0,1,0,1]_x s^2t^6$.

- $-2[2,0,0,1]_x s^2 t^6$ *relations.* There are precisely two distinct generator products at this order which are

$$
\begin{aligned}
{O_{(II)}}^{ij}{}_k &= {A_{001}}^{mni}{A_{001}}^{pqj}\epsilon_{mnpqk} = 0\,, \\
{O_{(III)}}^{ij}{}_k &= {A_{002}}^{imj}B_{mk} = 0\,,
\end{aligned}
\tag{5.156}
$$

which individually exactly vanish. These are the two quadratic relations which transform as $-[2,0,0,1]_x s^2 t^6$.

- $-[0,0,2,0]_x s^2 t^6$ *relations.* The following generator products correspond to the order $-[0,0,2,0]_x s^2 t^6$,

$$
\begin{aligned}
{o_{(I)}}_{ijkl} &= \epsilon_{pqsij}\epsilon_{mnrkl}{A_{001}}^{pqr}{A_{001}}^{mns}\,, \\
{o_{(II)}}_{ijkl} &= u_2 B_{ij}B_{kl}\,,
\end{aligned}
\tag{5.157}
$$

which satisfy

$$
\begin{aligned}
{o_{(I)}}_{ijkl} = -{o_{(I)}}_{jikl}\,, {o_{(I)}}_{ijkl} = -{o_{(I)}}_{ijlk}\,, {o_{(I)}}_{ijkl} = {o_{(I)}}_{jilk}\,, \\
{o_{(II)}}_{ijkl} = -{o_{(II)}}_{jikl}\,, {o_{(II)}}_{ijkl} = -{o_{(II)}}_{ijlk}\,, {o_{(II)}}_{ijkl} = {o_{(II)}}_{jilk}\ .
\end{aligned}
\tag{5.158}
$$

The above generator products satisfy the following quadratic relation,

$$
{O_{(IV)}}_{ijkl} = {o_{(I)}}_{ijkl} - \frac{1}{36}{o_{(II)}}_{ijkl} = 0\,,
\tag{5.159}
$$

which is the relations corresponding to $-[0,0,2,0]_x s^2 t^6$.

- $-[1,1,1,0]_x s^2 t^6$ *relations.* At this order, we can consider the following generator products,

$$
\begin{aligned}
{o_{(III)}}^{ijk}{}_{mn} &= {A_{001}}^{pqk}{A_{001}}^{ijr}\epsilon_{pqrmn}\,, \\
{o_{(IV)}}^{ijk}{}_{mn} &= {A_{002}}^{ijk}B_{mn}\,,
\end{aligned}
\tag{5.160}
$$

which satisfy the following quadratic relation,

$$
{O_{(V)}}^{ijk}{}_{mn} = {o_{(III)}}^{ijk}{}_{mn} - \frac{1}{3}{o_{(IV)}}^{ijk}{}_{mn} = 0\,.
\tag{5.161}
$$

This is precisely the relations transforming in $-[1,1,1,0]_x s^2 t^6$.

- $-[1,0,0,0]_x s^2 t^6$ *relations.* The following generator product vanishes exactly,

$$
{O_{(VI)}}^{i} = {A_{002}}^{mni}B_{mn} = 0\,,
\tag{5.162}
$$

which is the quadratic relation at the order $-[1,0,0,0]_x s^2 t^6$.

For the orders containing s^3t^6, we consider the following products of SU(5) representations,

$$\begin{aligned}[1,1,0,0]_x \times [1,1,0,0]_x &= [2,2,0,0]_x + [3,0,1,0]_x + [0,3,0,0]_x + 2[1,1,1,0]_x \\ &\quad +[2,0,0,1] + [0,0,2,0]_x + [0,1,0,1]_x\,, \\ [3,0,0,0]_x \times [0,0,1,0]_x &= [3,0,1,0]_x + [2,0,0,1]_x\,, \\ [1,1,0,0]_x \times [0,0,1,0]_x &= [1,1,1,0]_x + [2,0,0,1]_x + [0,1,0,1]_x + [1,0,0,0]_x\,, \\ \mathrm{Sym}^2[0,0,1,0]_x &= [0,0,2,0]_x + [1,0,0,0]_x\,. \end{aligned} \tag{5.163}$$

The above products are used to construct the following quadratic relations corresponding to order containing s^3t^6:

- $-[0,1,0,1]_x s^3t^6$ *relations.* At this order, we have the following generator product

$$P_{(I)}{}^{ij}{}_k = A_{001}{}^{ijm} A_{002}{}^{pqr} \epsilon_{mpqrk} = 0\,, \tag{5.164}$$

  which exactly vanishes. It corresponds to the quadratic relation for this order.

- $-2[2,0,0,1]_x s^3t^6$ *relations.* The following two generator product correspond to this order,

$$P_{(II)}{}^{ij}{}_k = A_{001}{}^{mni} A_{002}{}^{pqj} \epsilon_{mnpqk} = 0\,, \tag{5.165}$$

$$P_{(III)}{}^{ij}{}_k = S_{012}{}^{ijm} B_{mk} = 0\,, \tag{5.166}$$

  and exactly vanish. These are precisely the quadratic relations at this order.

- $-[0,0,2,0]_x s^3t^6$ *relations.* At this order, we consider the following generator products,

$$\begin{aligned} p_{(I)}{}_{ijkl} &= A_{001}{}^{pqm} A_{002}{}^{rsn} \epsilon_{pqnij} \epsilon_{rsmkl}\,, \\ p_{(II)}{}_{ijkl} &= u_3 B_{ij} B_{kl}\,, \end{aligned} \tag{5.167}$$

  which satisfy the following quadratic relation,

$$P_{(IV)}{}_{ijkl} = p_{(I)}{}_{ijkl} - \frac{1}{9} p_{(II)}{}_{ijkl} = 0\,. \tag{5.168}$$

  The above is precisely the quadratic relation at the order $-[0,0,2,0]_x s^3t^6$.

- $-2[1,1,1,0]_x s^3t^6$ *relations.* The following generator products can be considered at this order,

$$\begin{aligned} p_{(III)}{}^{ijk}{}_{mn} &= A_{001}{}^{ijp} A_{002}{}^{qrk} \epsilon_{pqrmn}\,, \\ p_{(IV)}{}^{ijk}{}_{mn} &= A_{002}{}^{ijp} A_{001}{}^{qrk} \epsilon_{pqrmn}\,, \\ p_{(V)}{}^{ijk}{}_{mn} &= u_2 A_{001}{}^{ijk} B_{mn}\,, \\ p_{(VI)}{}^{ijk}{}_{mn} &= u_3 \epsilon^{ijkpq} B_{pq} B_{mn}\,, \end{aligned} \tag{5.169}$$

which satisfy the following quadratic relations,

$$P_{(V)}{}^{ijk}{}_{mn} = p_{(III)}{}^{ijk}{}_{mn} - p_{(IV)}{}^{ijk}{}_{mn} = 0\,, \tag{5.170}$$
$$P_{(VI)}{}^{ijk}{}_{mn} = p_{(III)}{}^{ijk}{}_{mn} - \frac{1}{6}p_{(V)}{}^{ijk}{}_{mn} - \frac{1}{108}p_{(VI)}{}^{ijk}{}_{mn} = 0\,. \tag{5.171}$$

The above quadratic relations transform in the correct representation of this order, and hence are the relations we are looking for.

- $-[3,0,1,0]_x s^3 t^6$ *relations.* For this order, we have to consider the following generator products,

$$\begin{aligned} p_{(VII)}{}^{ijk}{}_{mn} &= (A_{001}{}^{ipj}A_{002}{}^{qrk} + A_{001}{}^{ipj}A_{002}{}^{qrk} + A_{001}{}^{ipj}A_{002}{}^{qrk})\epsilon_{pqrmn}\,, \\ p_{(VIII)}{}^{ijk}{}_{mn} &= S_{012}{}^{ijk}B_{nm}\,, \end{aligned} \tag{5.172}$$

where for the first product above we have symmetrized the product

$$A_{001}{}^{ipj}A_{002}{}^{qrk} \tag{5.173}$$

in the indices ijk. The products above satisfy the following quadratic relation,

$$P_{(VII)}{}^{ijk}{}_{mn} = p_{(VII)}{}^{ijk}{}_{mn} + \frac{1}{3}p_{(VIII)}{}^{ijk}{}_{mn} = 0\,. \tag{5.174}$$

The above is precisely the quadratic relation at this order.

The next orders contain s^4t^6 for which we have to consider the following representation products,

$$\begin{aligned} \mathrm{Sym}^2[1,1,0,0]_x &= [2,2,0,0]_x + [1,1,1,0]_x + [2,0,0,1]_x + [0,0,2,0]_x\,, \\ [1,1,0,0]_x \times [3,0,0,0]_x &= [4,1,0,0]_x + [2,2,0,0]_x + [3,0,1,0]_x + [1,1,1,0]_x\,, \\ [1,1,0,0]_x \times [0,0,1,0]_x &= [1,1,1,0]_x + [2,0,0,1]_x + [0,1,0,1]_x + [1,0,0,0]_x\,, \\ \mathrm{Sym}^2[0,0,1,0]_x &= [0,0,2,0]_x + [1,0,0,0]_x\,. \end{aligned} \tag{5.175}$$

The above representation products lead to the following quadratic relations of vortex moduli space generators:

- $-[2,0,0,1]_x s^4 t^6$ *relations.* The following generator product corresponds to the order $-[2,0,0,1]_x s^4 t^6$,

$$U_{(I)}{}^{ij}{}_{k} = A_{002}{}^{pqi}A_{002}{}^{mnj}\epsilon_{pqmsk} = 0\,, \tag{5.176}$$

which vanishes exactly. This is exactly the quadratic relation at this order.

- $-[0,0,2,0]_x s^4 t^6$ *relations.* This order refers to the following generator products,

$$\begin{aligned} u_{(I)}{}_{ijkl} &= A_{002}{}^{pqr}A_{002}{}^{mns}\epsilon_{pqsij}\epsilon_{mnrkl}\,, \\ u_{(II)}{}_{ijkl} &= u_2u_2B_{ij}B_{kl}\,, \end{aligned} \tag{5.177}$$

which satisfy then following relation,

$$U_{(II)}{}_{ijkl} = u_{(I)}{}_{ijkl} - \frac{1}{18}u_{(II)}{}_{ijkl} = 0\,. \tag{5.178}$$

This is precisely the quadratic relation corresponding to the order $-[0,0,2,0]_x s^4 t^6$.

- $-2[1,1,1,0]_x s^4 t^6$ *relations.* At this order, we need to consider the following generator products,

$$\begin{aligned}
u_{(III)}{}^{ijk}{}_{mn} &= A_{002}{}^{ijp} A_{002}{}^{qrk} \epsilon_{pqrmn}\,, \\
u_{(IV)}{}^{ijk}{}_{mn} &= A_{001}{}^{ijp} S_{012}{}^{qrk} \epsilon_{pqrmn}\,, \\
u_{(V)}{}^{ijk}{}_{mn} &= u_2 A_{001}{}^{ijp} A_{001}{}^{qrk} \epsilon_{pqrmn}\,, \\
u_{(VI)}{}^{ijk}{}_{mn} &= u_3 A_{001}{}^{ijk} B_{mn}\,.
\end{aligned} \tag{5.179}$$

The above products satisfy the following quadratic relations,

$$\begin{aligned}
U_{(III)}{}^{ijk}{}_{mn} &= u_{(V)}{}^{ijk}{}_{mn} - 2u_{(III)}{}^{ijk}{}_{mn} + \frac{2}{9} u_{(VI)}{}^{ijk}{}_{mn} = 0\,, \\
U_{(IV)}{}^{ijk}{}_{mn} &= u_{(IV)}{}^{ijk}{}_{mn} + \frac{1}{6} u_{(VI)}{}^{ijk}{}_{mn} = 0\,,
\end{aligned} \tag{5.180}$$

which are precisely the relations corresponding to the order $-2[1,1,1,0]_x s^4 t^6$.

- $-[2,2,0,0]_x s^4 t^6$ *relations.* For the quadratic relation at this order we have to consider the following generator product and additional symmetrization and anti-symmetrizations of its indices,

$$u_{(V)}{}^{ijklmn} = A_{001}{}^{ijk} S_{012}{}^{lmn}\,. \tag{5.181}$$

We anti-symmetrize first the indices $[kl]$ and $[mn]$ as follows,

$$u_{(VI)}{}^{ijklmn} = u_{(V)}{}^{ijklmn} - u_{(V)}{}^{ijlkmn} - u_{(V)}{}^{ijklnm} + u_{(V)}{}^{ijlknm}\,, \tag{5.182}$$

and symmetrize the pairs of indices $[kl]$ and $[mn]$ to give

$$U_{(V)}{}^{ijklmn} = u_{(VI)}{}^{ijklmn} + u_{(VI)}{}^{ijmnkl} = 0\,. \tag{5.183}$$

The above vanishes non-trivially, and given that the above quadratic relation transforms in the correct representation we have found the quadratic relation at this order.

- $-[3,0,1,0]_x s^4 t^6$ *relations.* For this order, we consider the following generator product,

$$U_{(VI)}{}^{ijk}{}_{mn} = (A_{001}{}^{ipq} S_{012}{}^{rjk} + A_{001}{}^{jpq} S_{012}{}^{rki} + A_{001}{}^{kpq} S_{012}{}^{rij}) \epsilon_{pqrmn} = 0\,, \tag{5.184}$$

which exactly vanishes. This is precisely the quadratic relation at this order. We have in above symmetrized the product

$$A_{001}{}^{ipq} S_{012}{}^{rjk} \tag{5.185}$$

in the indices ijk.

The next set of quadratic relations refer to orders containing s^5t^6. For these relations, we consider the following representation products,

$$\begin{aligned}
[3,0,0,0]_x \times [1,1,0,0]_x &= [4,1,0,0]_x + [2,2,0,0]_x + [3,0,1,0]_x + [1,1,1,0]_x\,, \\
[1,1,0,0]_x \times [1,1,0,0]_x &= [2,2,0,0]_x + [3,0,1,0]_x + [0,3,0,0]_x + 2[1,1,1,0]_x \\
&\quad + [2,0,0,1]_x + [0,0,2,0]_x + [0,1,0,1]_x\,, \\
[3,0,0,0]_x \times [0,0,1,0]_x &= [3,0,1,0]_x + [2,0,0,1]_x\,, \\
[1,1,0,0]_x \times [0,0,1,0]_x &= [1,1,1,0]_x + [2,0,0,1]_x + [0,1,0,1]_x + [1,0,0,0]_x\,, \\
\mathrm{Sym}^2[1,1,0,0]_x &= [2,2,0,0]_x + [1,1,1,0]_x + [2,0,0,1]_x + [0,0,2,0]_x\,, \\
\mathrm{Sym}^2[0,0,1,0]_x &= [0,0,2,0]_x + [1,0,0,0]_x\,.
\end{aligned} \tag{5.186}$$

The above products are used to identify the following quadratic relations between moduli space generators:

- $-[1,1,1,0]_x s^5t^6$ *relations.* For the order $-[1,1,1,0]_x s^5t^6$, we consider the following generator products,

$$\begin{aligned}
v_{(I)}{}^{ijk}{}_{mn} &= A_{002}{}^{ijp} S_{012}{}^{qrk} \epsilon_{pqrmn}\,, \\
v_{(II)}{}^{ijk}{}_{mn} &= u_3 A_{001}{}^{ijp} A_{001}{}^{qrk} \epsilon_{pqrmn}\,.
\end{aligned} \tag{5.187}$$

  Together the above products satisfy the following quadratic relations

$$V_{(I)}{}^{ijk}{}_{mn} = v_{(I)}{}^{ijk}{}_{mn} + \frac{1}{2} v_{(II)}{}^{ijk}{}_{mn} = 0\,, \tag{5.188}$$

  which is precisely the relation we are looking for at order $-[1,1,1,0]_x s^5t^6$.

- $-[2,2,0,0]_x s^5t^6$ *relations.* For the quadratic relation at this order, we have to consider the following generator products and its symmetrization and anti-symmetrization of indices. The first product to consider is the following,

$$v_{(III)}{}^{ijklmn} = (A_{002}{}^{ijk} - \frac{1}{2} u_2 B^{ijk}) S_{012}{}^{lmn}\,, \tag{5.189}$$

  which we anti-symmetrize in the indices $[kl]$ and $[mn]$ to give

$$v_{(IV)}{}^{ijklmn} = v_{(III)}{}^{ijklmn} - v_{(III)}{}^{ijlkmn} - v_{(III)}{}^{ijklnm} + v_{(III)}{}^{ijlknm}\,. \tag{5.190}$$

  We further anti-symmetrize the above in the pairs of indices $[ij]$ and $[mn]$ to give

$$v_{(V)}{}^{ijklmn} = v_{(IV)}{}^{ijklmn} - v_{(IV)}{}^{mnklij}\,. \tag{5.191}$$

  The second generator product to consider is the following

$$v_{(VI)}{}^{ijklmn} = u_3 A_{001}{}^{ijk} A_{001}{}^{lmn}\,, \tag{5.192}$$

  which is anti-symmetrized in the indices $[kl]$ and $[mn]$ as follows

$$v_{(VII)}{}^{ijklmn} = v_{(VI)}{}^{ijklmn} - v_{(VI)}{}^{ijlkmn} - v_{(VI)}{}^{ijklnm} + v_{(VI)}{}^{ijlknm}\,. \tag{5.193}$$

In addition, we symmetrize the pairs of indices $[ij]$ and $[kl]$ to give

$$v_{(VIII)}{}^{ijklmn} = v_{(VII)}{}^{ijklmn} + v_{(VII)}{}^{klijmn} \,. \tag{5.194}$$

The above generator products transform in the correct representation at this order of the plethystic logarithm, and satisfy the following quadratic relation

$$V_{(II)}^{ijklmn} = v_{(V)}{}^{ijklmn} + v_{(VIII)}{}^{ijmnkl} = 0. \tag{5.195}$$

The above is the quadratic relation we are searching for at this order.

- $-[3,0,1,0]_x s^5 t^6$ *relations.* For this order, we have to consider,

$$\begin{aligned} v_{(IX)}{}^{ijk}{}_{mn} &= (A_{002}{}^{ipq} S_{012}{}^{rjk} + A_{002}{}^{jpq} S_{012}{}^{rki} + A_{002}{}^{kpq} S_{012}{}^{rij}) \epsilon_{pqrmn} \,, \\ v_{(X)}{}^{ijk}{}_{mn} &= u_2 S_{012}{}^{ijk} B_{mn} \,, \end{aligned} \tag{5.196}$$

where above the generator product

$$A_{002}{}^{ipq} S_{012}{}^{rjk} \epsilon_{pqrmn} \,, \tag{5.197}$$

have been symmetrised in the indices ijk. The above satisfy the following quadratic relation,

$$V_{(III)}{}^{ijk}{}_{mn} = v_{(IX)}{}^{ijk}{}_{mn} - \frac{1}{3} v_{(X)}{}^{ijk}{}_{mn} = 0 \,, \tag{5.198}$$

which is the relation at order $-[3,0,1,0]_x s^5 t^6$.

The following order of $s^6 t^6$ leads us to use the following SU(5) representation products,

$$\begin{aligned} \mathrm{Sym}^2[3,0,0,0]_x &= [6,0,0,0]_x + [2,2,0,0]_x \,, \\ \mathrm{Sym}^2[1,1,0,0]_x &= [2,2,0,0]_x + [1,1,0,0]_x + [2,0,0,1]_x + [0,0,2,0]_x \,, \\ [1,1,0,0]_x \times [3,0,0,0]_x &= [4,1,0,0]_x + [2,2,0,0]_x + [3,0,1,0]_x + [1,1,1,0]_x \,, \\ [1,1,0,0]_x \times [1,1,0,0]_x &= [2,2,0,0]_x + [3,0,1,0]_x + [0,3,0,0]_x + 2[1,1,1,0]_x \\ &\quad + [2,0,0,1]_x + [0,0,2,0]_x + [0,1,0,1]_x \,. \end{aligned} \tag{5.199}$$

From the above representation products we select the appropriate one to identify the following quadratic relations between generators:

- $-[2,2,0,0]_x s^6 t^6$ *relations.* The quadratic relation at this order is constructed from the following generator product

$$z_{(I)}{}^{ijklmn} = S_{012}{}^{ijk} S_{012}{}^{lmn} \,, \tag{5.200}$$

which we first anti-symmetrize in the indices $[kl]$ and $[mn]$ as follows,

$$z_{(II)}{}^{ijklmn} = z_{(I)}{}^{ijklmn} - z_{(I)}{}^{ijlkmn} - z_{(I)}{}^{ijklnm} + z_{(I)}{}^{ijlknm} \,. \tag{5.201}$$

Then, we symmetrize the pairs of indices $[kl]$ and $[mn]$ such that

$$Z^{ijklmn} = z_{(II)}{}^{ijklmn} + z_{(II)}{}^{ijmnkl} = 0 \,, \tag{5.202}$$

vanishes exactly and hence is the quadratic relation we are looking for at this order.

Vortex moduli space. The $\mathbb{C}^*$ projection of the vortex master space $\mathcal{F}^{\flat}_{3,5}$ gives the full moduli space $\widetilde{\mathcal{V}}_{3,5}$ of the 3 U(5) vortex theory. The vortex moduli space is expressed as the following $\mathbb{C}^*$ projection,

$$
\begin{aligned}
\widetilde{\mathcal{V}}_{3,5} = \widetilde{\mathcal{F}^{\flat}}_{3,5}/\{B_{ij} \simeq \lambda^3 B_{ij}, {A_{001}}^{ijk} \simeq \lambda^3 {A_{001}}^{ijk},& \\
{A_{002}}^{ijk} \simeq \lambda^3 {A_{002}}^{ijk}, {S_{012}}^{ijk} \simeq \lambda^3 {S_{012}}^{ijk}\}\,,&
\end{aligned} \tag{5.203}
$$

where λ is the $\mathbb{C}^*$ parameter. The master space is expressed as follows,

$$
\begin{aligned}
\widetilde{\mathcal{F}^{\flat}}_{3,5} = \mathbb{C}[u_2, u_3, B_{ij}, {A_{001}}^{ijk}, {A_{002}}^{ijk}, {S_{012}}^{ijk}]/\{ & \\
& H^i = 0\,, \\
& {R_{(I)}}^{ij}{}_k = 0\,, {R_{(II)}}^{ij}{}_k = 0\,, {R_{(III)}}^{i} = 0\,, \\
& {O_{(I)}}^{ij}{}_k = 0\,, {O_{(II)}}^{ij}{}_k = 0\,, {O_{(III)}}^{ij}{}_k = 0\,, O_{(IV)ijkl} = 0\,, \\
& {O_{(V)}}^{ijk}{}_{mn} = 0\,, {O_{(VI)}}^{i} = 0\,, \\
& {P_{(I)}}^{ij}{}_k = 0\,, {P_{(II)}}^{ij}{}_k = 0\,, {P_{(III)}}^{ij}{}_k = 0\,, P_{(IV)ijkl} = 0\,, \\
& {P_{(V)}}^{ijk}{}_{mn} = 0\,, {P_{(VI)}}^{ijk}{}_{mn} = 0\,, {P_{(VII)}}^{ijk}{}_{mn} = 0\,, \\
& {U_{(I)}}^{ij}{}_k = 0\,, U_{(II)ijkl} = 0\,, {U_{(III)}}^{ijk}{}_{mn} = 0\,, {U_{(IV)}}^{ijk}{}_{mn} = 0\,, \\
& {U_{(V)}}^{ijklmn} = 0\,, {U_{(VI)}}^{ijk}{}_{mn} = 0\,, \\
& {V_{(I)}}^{ijk}{}_{mn} = 0\,, {V_{(II)}}^{ijklmn} = 0\,, {V_{(III)}}^{ijk}{}_{mn} = 0\,, \\
& Z^{ijklmn} = 0 \quad \}\,.
\end{aligned} \tag{5.204}
$$

5.6 3 U(6) vortices on $\mathbb{C}$

The Hilbert series for the 3 U(6) vortex master space is given by the Molien integral,

$$
g(t,s,x;\widetilde{\mathcal{F}^{\flat}}_{3,6}) = \oint d\mu_{\mathrm{SU}(3)} \mathrm{PE}\Big[[0,1]_w[1,0,0,0,0]_x + [1,1]_w s\Big]\,, \tag{5.205}
$$

where $[1,0,0,0,0]_w$ is the fundamental representation of the global SU(6). The integral leads to the following character expansion of the Hilbert series

$$
\begin{aligned}
g(t,s,x;\widetilde{\mathcal{F}^{\flat}}_{3,6}) = & \frac{1}{(1-s^2)(1-s^3)} \times \\
\sum_{n_0=0}^{\infty}\sum_{n_1=0}^{\infty}\sum_{n_2=0}^{\infty}\sum_{n_3=0}^{\infty} & \Big[[n_1+n_2+3n_3, n_1+n_2, n_0, 0, 0]_x s^{n_1+2n_2+3n_3} t^{3n_0+3n_1+3n_2+3n_3} \\
& + [n_1+n_2, n_1+n_2+3n_3+3, n_0, 0, 0]_x s^{n_1+2n_2+3n_3+3} t^{3n_0+3n_1+3n_2+6n_3+6}\Big]\,,
\end{aligned} \tag{5.206}
$$

where $[m_1, m_2, m_3, m_4]_x$ is a character of a SU(5) irreducible representation.

The plethystic logarithm of the Hilbert series is

$$\begin{aligned}\mathrm{PL}\Big[g(t,s,x;\widetilde{\mathcal{F}^\flat}_{3,6})\Big] = {} & s^2+s^3+[0,0,1,0,0]_x t^3+[1,1,0,0,0]_x s t^3+[1,1,0,0,0]_x s^2t^3\\ & +[3,0,0,0,0]_x s^3t^3-[1,0,0,0,1]_x t^6-([2,0,0,1,0]_x\\ & +[0,1,0,1,0]_x+[1,0,0,0,1]_x)st^6-([1,1,1,0,0]_x+[0,0,2,0,0]_x\\ & +[2,0,0,1,0]_x+[0,1,0,1,0]_x+[1,0,0,0,1]_x)s^2t^6\\ & +([1,1,0,0,0]_x+[0,0,0,1,1]_x)t^9-([3,0,1,0,0]_x\\ & +2[1,1,1,0,0]_x+[0,0,2,0,0]_x+2[2,0,0,1,0]_x+[0,1,0,1,0]_x\\ & +[1,0,0,0,1]_x)s^3t^6+\dots-([0,0,2,0,0]_x+[1,1,1,0,0]_x\\ & +[2,0,0,1,0]_x+[2,2,0,0,0]_x+[3,0,1,0,0]_x)s^4t^6+\dots\\ & -([1,1,1,0,0]_x+[2,2,0,0,0]_x+[3,0,1,0,0]_x)s^5t^6+\dots\\ & -[2,2,0,0,0]_x s^6t^6+\dots\,. \end{aligned} \tag{5.207}$$

The generators of the vortex moduli space are indicated by the above plethystic logarithm. They are as follows,

$$\begin{aligned} s^2 &\to u_2=\mathrm{Tr}(\phi^2)\\ s^3 &\to u_3=\mathrm{Tr}(\phi^3)\\ [0,0,1,0,0]_x t^3 &\to B^{ijk}=\epsilon^{\alpha_1\alpha_2\alpha_3}Q^i_{\alpha_1}Q^j_{\alpha_2}Q^k_{\alpha_3}\\ [1,1,0,0,0]_x st^3 &\to \begin{cases} {A_{001}}^{ijk}=\epsilon^{\alpha_1\alpha_2\alpha_3}Q^i_{\alpha_1}Q^j_{\alpha_2}\phi^{\beta}_{\alpha_3}Q^k_{\beta}\\ \epsilon_{ijkmnp}{A_{001}}^{ijk}=0\end{cases}\\ [1,1,0,0,0]_x s^2t^3 &\to \begin{cases} {A_{002}}^{ijk}=\epsilon^{\alpha_1\alpha_2\alpha_3}Q^i_{\alpha_1}Q^j_{\alpha_2}\phi^{\beta_1}_{\alpha_3}\phi^{\beta_2}_{\beta_1}Q^k_{\beta_2}\\ \epsilon_{ijkmnp}{A_{002}}^{ijk}=-\frac{1}{3}u_2\epsilon_{mnprsu}B^{rsu}\\ {A_{011}}^{ijk}=\epsilon^{\alpha_1\alpha_2\alpha_3}Q^k_{\alpha_1}\phi^{\beta_1}_{\alpha_2}Q^i_{\beta_1}\phi^{\beta_2}_{\alpha_3}Q^j_{\beta_2}\\ \epsilon_{ijkmnp}{A_{011}}^{ijk}=\frac{1}{6}u_2\epsilon_{mnprsu}B^{rsu}\\ \to {A_{002}}^{ijk}={A_{011}}^{ijk}+\frac{1}{2}u_2B^{ijk}\end{cases}\\ [3,0,0,0,0]_x s^3t^3 &\to \begin{cases} {A_{012}}^{ijk}=\epsilon^{\alpha_1\alpha_2\alpha_3}Q^i_{\alpha_1}\phi^{\beta_1}_{\alpha_2}Q^j_{\beta_1}\phi^{\beta_2}_{\alpha_3}\phi^{\beta_3}_{\beta_2}Q^k_{\beta_3}\\ {S_{012}}^{ijk}={A_{012}}^{ijk}+{A_{012}}^{jki}+{A_{012}}^{kij}\end{cases}\quad. \end{aligned}$$

Quadratic relations. The terms of the plethystic logarithm corresponding to quadratic relations between master space generators are as follows,

$$\begin{gathered} -[1,0,0,0,1]_x t^6\\ -[0,1,0,1,0]_x st^6-[2,0,0,1,0]_x st^6-[1,0,0,0,1]_x st^6\\ -[0,1,0,1,0]_x s^2t^6-2[2,0,0,1,0]_x s^2t^6-[0,0,2,0,0]_x s^2t^6\\ -[1,1,1,0,0]_x s^2t^6-[1,0,0,0,1]_x s^2t^6\\ -[0,1,0,1,0]_x s^3t^6-2[2,0,0,1,0]_x s^3t^6-[0,0,2,0,0]_x s^3t^6\\ -2[1,1,1,0,0]_x s^3t^6-[3,0,1,0,0]_x s^3t^6\\ -[2,0,0,1,0]_x s^4t^6-[0,0,2,0,0]_x s^4t^6-2[1,1,1,0,0]_x s^4t^6\\ -[2,2,0,0,0]_x s^4t^6-[3,0,1,0,0]_x s^4t^6\\ -[1,1,1,0,0]_x s^5t^6-[2,2,0,0,0]_x s^5t^6-[3,0,1,0,0]_x s^5t^6\\ -[2,2,0,0,0]_x s^6t^6\,. \end{gathered} \tag{5.208}$$

The quadratic relations are formed by the generators of the vortex master space which are as discussed above as follows,

$$u_2\,,u_3\,,B^{ijk}\,,{A_{001}}^{ijk}\,,{A_{002}}^{ijk}\,,{S_{012}}^{ijk}\,. \tag{5.209}$$

Let us consider the first quadratic relation at order t^6 which requires consideration of the following SU(6) representation product,

$$\mathrm{Sym}^2[0,0,1,0,0]_x=[0,0,2,0,0]_x+[1,0,0,0,1]_x\,. \tag{5.210}$$

The above symmetric product allows us to construct the following quadratic relation:

- $-[1,0,0,0,1]_x t^6$ *relations.* The following product of generators exactly vanishes,

$$H_i^{\ j}=\epsilon_{ipqrsu}B^{pqr}B^{suj}\,, \tag{5.211}$$

  and given that it transforms in the adjoint of SU(6) is exactly the quadratic relation at this order.

For the next set of quadratic relations containing the order st^6, we consider the following representation products,

$$\begin{aligned}[1,1,0,0,0]_x\times[0,0,1,0,0]_x=&\ [1,1,1,0,0]_x+[2,0,0,1,0]_x+[0,1,0,1,0]_x\\ &+[1,0,0,0,1]_x\,.\end{aligned} \tag{5.212}$$

The above tensor product allows us to construct the following quadratic relations:

- $-[0,1,0,1,0]_x st^6$ *relations.* At this order, we consider the following generator product,

$$R_{(I)}{}^{ij}{}_{kl}={A_{001}}^{ijm}B^{pqr}\epsilon_{mpqrkl}=0\,, \tag{5.213}$$

  which vanishes exactly. This transforms in the correct representation of SU(6) and hence is the quadratic relation at this order.

- $-[2,0,0,1,0]_x st^6$ *relations.* The following generator product vanishes exactly,

$$R_{(II)}{}^{ij}{}_{kl}={A_{001}}^{imj}B^{pqr}\epsilon_{mpqrkl}=0\,. \tag{5.214}$$

  It transforms in the correct representation and hence is the quadratic relation at this order.

- $-[1,0,0,0,1]_x st^6$ *relations.* For this order, the following generator product is considered,

$$R_{(III)}{}^{i}{}_{j}={A_{001}}^{imn}B^{klp}\epsilon_{mnklpj}=0\,. \tag{5.215}$$

  The above product exactly vanishes and since it is in the adjoint of SU(6) it is the quadratic relation at this order.

The next set of quadratic relations contains the order s^2t^6. For these relations, we consider the following representation products of SU(6),

$$\begin{aligned}\mathrm{Sym}^2[1,1,0,0,0]_x &= [2,2,0,0,0]_x + [1,1,1,0,0]_x + [2,0,0,1,0]_x + [0,0,2,0,0]_x\,,\\ [1,1,0,0,0]_x \times [0,0,1,0,0]_x &= [1,1,1,0,0]_x + [2,0,0,1,0]_x + [0,1,0,1,0]_x\\ &\quad +[1,0,0,0,1]_x\,,\\ \mathrm{Sym}^2[0,0,1,0,0]_x &= [0,0,2,0,0]_x + [1,0,0,0,1]_x\,. \end{aligned} \tag{5.216}$$

The following quadratic relations can be identified from the above representation products as follows:

- $-[0,1,0,1,0]_x s^2t^6$ *relations.* For this order, we consider the following generator product,

$$O_{(I)}{}^{ij}{}_{kl} = A_{002}{}^{ijp} B^{qrs} \epsilon_{pqrskl} = 0\,, \tag{5.217}$$

which exactly vanishes. The above transforms in the correct representation and therefore is precisely the quadratic relation for this order.

- $-2[2,0,0,1,0]_x s^2t^6$ *relations.* For the quadratic relations at this order, we consider the following generator products,

$$\begin{aligned} O_{(II)}{}^{ij}{}_{kl} &= A_{001}{}^{mni} A_{001}{}^{pqj} \epsilon_{mnpqkl} = 0\,,\\ O_{(III)}{}^{ij}{}_{kl} &= A_{002}{}^{imj} B^{npq} \epsilon_{mnpqkl} = 0\,, \end{aligned} \tag{5.218}$$

where the products above both vanish exactly. Given that they transform in the correct representation, we identify them as the 2 quadratic relations at this order.

- $-[0,0,2,0,0]_x s^2t^6$ *relations.* The following generator products are helpful in constructing the quadratic relations at this order,

$$\begin{aligned} o_{(I)}{}_{ijklmn} &= \epsilon_{pqsijk}\epsilon_{uvrlmn} A_{001}{}^{pqr} A_{001}^{uvs}\,,\\ o_{(II)}{}_{ijklmn} &= u_2 B^{pqs} B^{uvr} \epsilon_{pqsijk}\epsilon_{uvrlmn}\,. \end{aligned} \tag{5.219}$$

The above products transform in the representation of this order. They satisfy the following quadratic relation,

$$O_{(IV)}{}_{ijklmn} = o_{(I)}{}_{ijklmn} - \frac{1}{9} o_{(II)}{}_{ijklmn}\,, \tag{5.220}$$

which is precisely the relation at this order.

- $-[1,1,1,0,0]_x s^2t^6$ *relations.* For this order, we consider the following generator products,

$$\begin{aligned} o_{(III)}{}^{ijk}{}_{lmn} &= A_{001}{}^{pqk} A_{001}{}^{ijr} \epsilon_{pqrlmn}\,,\\ o_{(IV)}{}^{ijk}{}_{lmn} &= A_{002}{}^{ijk} B^{pqr} \epsilon_{pqrlmn}\,, \end{aligned} \tag{5.221}$$

which transform in the correct representation at this order. The above products satisfy the following quadratic relation,

$$O_{(V)}{}^{ijk}{}_{lmn} = o_{(III)}{}^{ijk}{}_{lmn} - \frac{1}{3} o_{(IV)}{}^{ijk}{}_{lmn} = 0\,, \tag{5.222}$$

which is precisely the relation at this order.

- $-[1,0,0,0,1]_x s^2 t^6$ *relations.* The following generator product vanishes exactly,

$$O_{(VI)}{}^{i}{}_{j} = A_{002}{}^{pqi} B^{lmn} \epsilon_{pqlmnj} = 0\,. \tag{5.223}$$

  The above quadratic relations transform in the adjoint of SU(6) and hence are the quadratic relations for this order.

For the next set of quadratic relations at orders containing s^3t^6, we first consider the following SU(6) representation products,

$$\begin{aligned}
[1,1,0,0,0]_x \times [1,1,0,0,0]_x &= [2,2,0,0,0]_x + [3,0,1,0,0]_x + [0,3,0,0,0]_x \\
&\quad +2[1,1,1,0,0]_x + [2,0,0,1,0]_x + [0,0,2,0,0]_x \\
&\quad +[0,1,0,1,0]_x\,, \\
[3,0,0,0,0]_x \times [0,0,1,0,0]_x &= [3,0,1,0,0]_x + [2,0,0,1,0]_x\,, \\
[1,1,0,0,0]_x \times [0,0,1,0,0]_x &= [1,1,1,0,0]_x + [2,0,0,1,0]_x + [0,1,0,1,0]_x \\
&\quad +[1,0,0,0,1]_x\,, \\
\mathrm{Sym}^2[0,0,1,0,0]_x &= [0,0,2,0,0]_x + [1,0,0,0,1]_x\,.
\end{aligned} \tag{5.224}$$

The above representation products help us in constructing the following quadratic relations between master space generators at order s^3t^6:

- $-[0,1,0,1,0]_x s^3 t^6$ *relations.* The following generator product vanishes exactly,

$$P_{(I)}{}^{ij}{}_{kl} = A_{001}{}^{ijm} A_{002}{}^{pqr} \epsilon_{mpqrkl} = 0\,, \tag{5.225}$$

  and transforms in the representation of this order. Accordingly, it is exactly the quadratic relation we are looking for this order.

- $-2[2,0,0,1,0]_x s^3 t^6$ *relations.* The following generator products are relevant for the quadratic relations at this order,

$$\begin{aligned}
P_{(II)}{}^{ij}{}_{kl} &= A_{001}{}^{mni} A_{002}{}^{pqj} \epsilon_{mnpqkl} = 0\,, \\
P_{(III)}{}^{ij}{}_{kl} &= S_{012}{}^{ijm} B^{npq} \epsilon_{mnpqkl} = 0\,.
\end{aligned} \tag{5.226}$$

  Both above vanish and satisfy the correct transformation property for this order. They are precisely the two quadratic relations at this order.

- $-[0,0,2,0,0]_x s^3 t^6$ *relations.* For this order, we consider the following generator products,

$$\begin{aligned} p_{(I)}{}_{ijklmn} &= A_{001}{}^{pqu} A_{002}{}^{rsv} \epsilon_{pqvijk} \epsilon_{rsulmn} \,, \\ p_{(II)}{}_{ijklmn} &= u_3 B^{pqu} B^{rsv} \epsilon_{pquijk} \epsilon_{rsvlmn} \,, \end{aligned} \tag{5.227}$$

which transform in the correct representation corresponding to this order. The above products satisfy the following quadratic relation

$$P_{(IV)}{}_{ijklmn} = p_{(I)}{}_{ijklmn} - \frac{1}{9} p_{(II)}{}_{ijklmn} \,, \tag{5.228}$$

which is the relation for this order.

- $-2[1,1,1,0,0]_x s^3 t^6$ *relations.* The following generator products are useful in constructing the quadratic relations for this order,

$$\begin{aligned} p_{(III)}{}^{ijk}{}_{lmn} &= A_{001}{}^{ijp} A_{002}{}^{qrk} \epsilon_{pqrlmn} \,, \\ p_{(IV)}{}^{ijk}{}_{lmn} &= A_{002}{}^{ijp} A_{001}{}^{qrk} \epsilon_{pqrlmn} \,, \\ p_{(V)}{}^{ijk}{}_{lmn} &= u_2 A_{001}{}^{ijp} B^{qrk} \epsilon_{pqrlmn} \,. \end{aligned} \tag{5.229}$$

The above products transform in the correct representation for this order. The quadratic relations formed by the above are

$$\begin{aligned} P_{(V)}{}^{ijk}{}_{lmn} &= p_{(III)}{}^{ijk}{}_{lmn} - p_{(IV)}{}^{ijk}{}_{lmn} = 0 \,, \\ P_{(VI)}{}^{ijk}{}_{lmn} &= p_{(III)}{}^{ijk}{}_{lmn} - \frac{1}{2} p_{(V)}{}^{ijk}{}_{lmn} = 0 \,, \end{aligned} \tag{5.230}$$

exactly corresponding to the two expected quadratic relations at this order.

- $-[3,0,1,0,0]_x s^3 t^6$ *relations.* For this order, we consider the following generator products,

$$\begin{aligned} p_{(VI)}{}^{ijk}{}_{lmn} &= (A_{001}{}^{ipj} A_{002}{}^{qrk} + A_{001}{}^{jpk} A_{002}{}^{qri} + A_{001}{}^{kpi} A_{002}{}^{qrj}) \epsilon_{pqrlmn} \,, \\ p_{(VII)}{}^{ijk}{}_{lmn} &= S_{012}{}^{ijk} B^{pqr} \epsilon_{pqrlmn} \,, \end{aligned} \tag{5.231}$$

which transform in the correct representation for this order. The above products satisfy the following quadratic relation

$$P_{(VII)}{}^{ijk}{}_{lmn} = p_{(VI)}{}^{ijk}{}_{lmn} + \frac{1}{3} p_{(VII)}{}^{ijk}{}_{lmn} \,, \tag{5.232}$$

which is precisely the relation for this order.

We can now consider the next set of quadratic generator relations which are at orders containing $s^4 t^6$. In order to construct the relations, we consider the following SU(6) representation products,

$$\begin{aligned}
\mathrm{Sym}^2[1,1,0,0,0]_x &= [2,2,0,0,0]_x+[1,1,1,0,0]_x+[2,0,0,1,0]_x+[0,0,2,0,0]_x\,,\\
[1,1,0,0,0]_x\times[3,0,0,0,0]_x &= [4,1,0,0,0]_x+[2,2,0,0,0]_x+[3,0,1,0,0]_x\\
&\quad +[1,1,1,0,0]_x\,,\\
[1,1,0,0,0]_x\times[0,0,1,0,0]_x &= [1,1,1,0,0]_x+[2,0,0,1,0]_x+[0,1,0,1,0]_x\\
&\quad +[1,0,0,0,1]_x\,,\\
\mathrm{Sym}^2[0,0,1,0,0]_x &= [0,0,2,0,0]_x+[1,0,0,0,1]_x\,. \qquad (5.233)
\end{aligned}$$

With the above representation products in mind, we construct the following quadratic relations:

- $-[2,0,0,1,0]_x s^4t^6$ *relations.* For this order, we consider the following generator product,

$$U_{(I)}{}^{ij}{}_{kl} = A_{002}{}^{pqi}A_{002}{}^{mnj}\epsilon_{pqmnkl}=0\,, \qquad (5.234)$$

  which vanishes exactly. This is precisely the quadratic relation at this order.

- $-[0,0,2,0,0]_x s^4t^6$ *relations.* In order to construct the quadratic relation for this order, we consider the following products,

$$\begin{aligned}
u_{(I)}{}_{ijklmn} &= A_{002}{}^{pqr}A_{002}{}^{uvs}\epsilon_{pqsijk}\epsilon_{uvrlmn}\,,\\
u_{(II)}{}_{ijklmn} &= u_2u_2B^{pqr}B^{uvw}\epsilon_{pqrijk}\epsilon_{uvwlmn}\,. \qquad (5.235)
\end{aligned}$$

  The above products transform in the correct SU(6) representation of this order. They satisfy the following quadratic relation,

$$U_{(II)}{}_{ijklmn} = u_{(I)}{}_{ijklmn}-\frac{1}{3}u_{(II)}{}_{ijklmn}=0\,, \qquad (5.236)$$

  which is precisely the relation we are looking for here.

- $-2[1,1,1,0,0]_x s^4t^6$ *relations.* Corresponding to this order, there are two distinct quadratic relations. In order to construct them, we consider the following generator products,

$$\begin{aligned}
u_{(III)}{}^{ijk}{}_{lmn} &= A_{002}{}^{ijp}A_{002}{}^{qrk}\epsilon_{pqrlmn}\,,\\
u_{(IV)}{}^{ijk}{}_{lmn} &= A_{001}{}^{ijp}S_{012}{}^{qrk}\epsilon_{pqrlmn}\,,\\
u_{(V)}{}^{ijk}{}_{lmn} &= u_2A_{001}{}^{ijk}A_{001}{}^{pqr}\epsilon_{pqrlmn}\,,\\
u_{(VI)}{}^{ijk}{}_{lmn} &= u_3A_{001}{}^{ijk}B^{pqr}\epsilon_{pqrlmn}\,, \qquad (5.237)
\end{aligned}$$

  which transform in the representation of this order. The above products form the following two quadratic relations,

$$\begin{aligned}
U_{(III)}{}^{ijk}{}_{lmn} &= u_{(V)}{}^{ijk}{}_{lmn}-2u_{(III)}{}^{ijk}{}_{lmn}+\frac{2}{9}u_{(VI)}{}^{ijk}{}_{lmn}=0\,,\\
U_{(IV)}{}^{ijk}{}_{lmn} &= u_{(IV)}{}^{ijk}{}_{lmn}+\frac{1}{6}u_{(VI)}{}^{ijk}{}_{lmn}=0\,. \qquad (5.238)
\end{aligned}$$

  The two quadratic relations above are precisely corresponding to this order.

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- $-[2,2,0,0,0]_x s^4t^6$ *relations.* The relation at this order requires us to have a look at the following product of generators,

$$u_{(VII)}{}^{ijklmn} = A_{001}{}^{ijk} S_{012}{}^{lmn}\,, \tag{5.239}$$

where we antisymmetrize on the indices $[kl]$ and $[mn]$ as follows,

$$u_{(VIII)}{}^{ijklmn} = u_{(VII)}{}^{ijklmn} - u_{(VII)}{}^{ijlkmn} - u_{(VII)}{}^{ijklnm} + u_{(VII)}{}^{ijlknm}\,. \tag{5.240}$$

A further symmetrization on the two paired indices $[kl]$ and $[mn]$ leads to the following

$$U_{(V)}{}^{ijklmn} = u_{(VIII)}{}^{ijklmn} + u_{(VIII)}{}^{ijmnkl} = 0 \tag{5.241}$$

which exactly vanishes. Given that the above expression is precisely of order $-[2,2,0,0,0]_x s^4t^6$, this is the quadratic relation of generators we are looking for.

- $-[3,0,1,0,0]_x s^4t^6$ *relations.* The quadratic relation at this order is formed by

$$U_{(VI)}{}^{ijk}{}_{lmn} = (A_{001}{}^{ipq} S_{012}{}^{rjk} + A_{001}{}^{jpq} S_{012}{}^{rki} + A_{001}{}^{jpq} S_{012}{}^{rij})\epsilon_{pqrlmn} = 0\,, \tag{5.242}$$

where the above contains the symmetrization of the generator product

$$A_{001}{}^{ipq} S_{012}{}^{rjk}\epsilon_{pqrlmn} \tag{5.243}$$

in the indices ijk. The above quadratic relation satisfies precisely the transformation properties for this order and is the relation we are looking for.

The next set of quadratic relation are at orders which contain s^5t^6. We consider the following representation products in order to construct the relations,

$$\begin{aligned}
[3,0,0,0,0]_x \times [1,1,0,0,0]_x &= [4,1,0,0,0]_x + [2,2,0,0,0]_x + [3,0,1,0,0]_x \\
&\quad +[1,1,1,0,0]_x\,, \\
[1,1,0,0,0]_x \times [1,1,0,0,0]_x &= [2,2,0,0,0]_x + [3,0,1,0,0]_x + [0,3,0,0,0]_x \\
&\quad +2[1,1,1,0,0]_x + [2,0,0,1,0]_x + [0,0,2,0,0]_x \\
&\quad +[0,1,0,1,0]_x\,, \\
[3,0,0,0,0]_x \times [0,0,1,0,0]_x &= [3,0,1,0,0]_x + [2,0,0,1,0]_x\,, \\
[1,1,0,0,0]_x \times [0,0,1,0,0]_x &= [1,1,1,0,0]_x + [2,0,0,1,0]_x + [0,1,0,1,0]_x \\
&\quad +[1,0,0,0,1]_x\,, \\
\mathrm{Sym}^2[1,1,0,0,0]_x &= [2,2,0,0,0]_x + [1,1,1,0,0]_x + [2,0,0,1,0]_x \\
&\quad +[0,0,2,0,0]_x\,, \\
\mathrm{Sym}^2[0,0,1,0,0]_x &= [0,0,2,0,0]_x + [1,0,0,0,1]_x\,.
\end{aligned} \tag{5.244}$$

The above products are used to construct the following quadratic relations of generators for vortex master spaces:

- $-[1,1,1,0,0]_x s^5 t^6$ *relations.* For the quadratic relation at this order, we consider the following products of generators,

$$\begin{aligned} v_{(I)}{}^{ijk}{}_{lmn} &= A_{002}{}^{ijp} S_{012}{}^{qrk} \epsilon_{pqrlmn}\,, \\ v_{(II)}{}^{ijk}{}_{lmn} &= u_3 A_{001}{}^{ijp} A_{001}{}^{qrk} \epsilon_{pqrlmn}\,, \end{aligned} \tag{5.245}$$

which transform in the correct irreducible representation of this order. The above products satisfy the following quadratic relation,

$$V_{(I)}{}^{ijk}{}_{lmn} = v_{(I)}{}^{ijk}{}_{lmn} + \frac{1}{2} v_{(II)}{}^{ijk}{}_{lmn} = 0\,. \tag{5.246}$$

The above precisely is the quadratic relation we are looking for at this order.

- $-[2,2,0,0,0]_x s^5 t^6$ *relations.* For the quadratic relation at this order, we need to consider several generator products with various symmetrizations and anti-symmetrizations of indices. The first product to consider is the following,

$$v_{(III)}{}^{ijklmn} = \left(A_{002}{}^{ijk} - \frac{1}{2} u_2 B^{ijk} \right) S_{012}{}^{lmn}\,, \tag{5.247}$$

where we recall that $A_{011}{}^{ijk} = A_{002}{}^{ijk} - \frac{1}{2} u_2 B^{ijk}$. Above, we anti-symmetrize in the indices $[kl]$ and $[mn]$ to give

$$v_{(IV)}{}^{ijklmn} = v_{(III)}{}^{ijklmn} - v_{(III)}{}^{ijlkmn} - v_{(III)}{}^{ijklnm} + v_{(III)}{}^{ijlknm}\,, \tag{5.248}$$

and further anti-symmetrize in the pairs of indices $[ij]$ and $[mn]$ to obtain,

$$v_{(V)}{}^{ijklmn} = v_{(IV)}{}^{ijklmn} - v_{(IV)}{}^{mnklij}\,. \tag{5.249}$$

The second generator product to consider is the following,

$$v_{(VI)}{}^{ijklmn} = u_3 A_{001}{}^{ijk} A_{001}{}^{lmn}\,, \tag{5.250}$$

which we anti-symmetrize in the indices $[kl]$ and $[mn]$ giving,

$$v_{(VII)}{}^{ijklmn} = v_{(VI)}{}^{ijklmn} - v_{(VI)}{}^{ijlkmn} - v_{(VI)}{}^{ijklnm} + v_{(VI)}{}^{ijlknm}\,. \tag{5.251}$$

A further symmetrization of the pair of indices $[ij]$ and $[kl]$ gives

$$v_{(VIII)}{}^{ijklmn} = v_{(VII)}{}^{ijklmn} + v_{(VII)}{}^{klijmn}\,. \tag{5.252}$$

From the above generator products, which correctly transform at this order, we can construct the following quadratic relation,

$$V_{(II)}{}^{ijklmn} = v_{(V)}{}^{ijklmn} + v_{(VIII)}{}^{ijmnkl} = 0\,. \tag{5.253}$$

This is precisely we are looking for at this order.

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- $-[3,0,1,0,0]_x s^5 t^6$ *relations.* The following generator products are of importance in order to construct the quadratic relation at this order. The first product is as follows

$$v_{(IX)}{}^{ijk}{}_{lmn} = A_{002}{}^{ipq} S_{012}{}^{rjk} \epsilon_{pqrlmn} \,, \tag{5.254}$$

which we symmetrize in the indices ijk as follows,

$$v_{(X)}{}^{ijk}{}_{lmn} = v_{(IX)}{}^{ijk}{}_{lmn} + v_{(IX)}{}^{jki}{}_{lmn} + v_{(IX)}{}^{kij}{}_{lmn} \,. \tag{5.255}$$

The second generator product which we need to consider is the following,

$$v_{(XI)}{}^{ijk}{}_{lmn} = u_2 S_{012}{}^{ijk} B^{pqr} \epsilon_{pqrlmn} \,. \tag{5.256}$$

From the above products, we construct the following quadratic relation,

$$V_{(III)}{}^{ijk}{}_{lmn} = v_{(X)}{}^{ijk}{}_{lmn} - \frac{1}{3} v_{(XI)}{}^{ijk}{}_{lmn} = 0 \,. \tag{5.257}$$

This is precisely the relation we are looking for at this order of the plethystic logarithm.

The next order of $s^6 t^6$ leads us to the following products of SU(6) representations,

$$\begin{aligned}
\mathrm{Sym}^2[3,0,0,0,0]_x &= [6,0,0,0,0]_x + [2,2,0,0,0]_x \,, \\
\mathrm{Sym}^2[1,1,0,0,0]_x &= [2,2,0,0,0]_x + [1,1,1,0,0]_x + [2,0,0,1,0]_x \\
&\quad + [0,0,2,0,0]_x \,, \\
[1,1,0,0,0]_x \times [3,0,0,0,0]_x &= [4,1,0,0,0]_x + [2,2,0,0,0]_x + [3,0,1,0,0]_x \\
&\quad + [1,1,1,0,0]_x \,, \\
[1,1,0,0,0]_x \times [1,1,0,0,0]_x &= [2,2,0,0,0]_x + [3,0,1,0,0]_x + [0,3,0,0,0]_x \\
&\quad + 2[1,1,1,0,0]_x + [2,0,0,1,0]_x + [0,0,2,0,0]_x \\
&\quad + [0,1,0,1,0]_x \,.
\end{aligned} \tag{5.258}$$

From the above, we can construct candidate products of generators of the vortex master space, in order to identify quadratic relations amongst them:

- $-[2,2,0,0,0]_x s^6 t^6$ *relations.* For the quadratic relations at this order, we consider the following product of generators of the vortex master space,

$$z_{(I)}{}^{ijklmn} = S_{012}{}^{ijk} S_{012}{}^{lmn} \,. \tag{5.259}$$

We anti-symmetrize the above on the indices $[kl]$ and $[mn]$

$$z_{(II)}{}^{ijklmn} = z_{(I)}{}^{ijklmn} - z_{(I)}{}^{ijlkmn} - z_{(I)}{}^{ijklnm} + z_{(I)}{}^{ijlknm} \,, \tag{5.260}$$

and further symmetrize on the pairs of indices $[kl]$ and $[mn]$ to give

$$Z^{ijklmn} = z_{(II)}{}^{ijklmn} + z_{(II)}{}^{ijmnkl} = 0 \,. \tag{5.261}$$

The above quadratic relation vanishes non-trivially and by construction transforms in the representation of this order of the plethystic logarithm. The above is the quadratic relation corresponding to this order.

Vortex moduli space. Given the above explicit computation of the quadratic relations between generators, we can proceed in identifying the vortex moduli space for the 3 U(6) vortex theory. The vortex moduli space can be expressed as a $\mathbb{C}^*$ projection as follows,

$$\begin{aligned}\widetilde{\mathcal{V}}_{3,6} = \widetilde{\mathcal{F}^\flat}_{3,6}/\{&B^{ijk} \simeq \lambda^3 B^{ijk}, {A_{001}}^{ijk} \simeq \lambda^3 {A_{001}}^{ijk}, \\ &{A_{002}}^{ijk} \simeq \lambda^3 {A_{002}}^{ijk}, {S_{012}}^{ijk} \simeq \lambda^3 {S_{012}}^{ijk}\}\,. \end{aligned} \tag{5.262}$$

The master space of the vortex theory can be identified with the help of the quadratic relations as follows,

$$\begin{aligned}\widetilde{\mathcal{F}^\flat}_{3,6} = \mathbb{C}[u_2, u_3, B^{ijk}, {A_{001}}^{ijk}, {A_{002}}^{ijk}, {S_{012}}^{ijk}]/\{ \\ &{H_i}^j = 0\,, \\ &{{R_{(I)}}^{ij}}_{kl} = 0\,, {{R_{(II)}}^{ij}}_{kl} = 0\,, {{R_{(III)}}^{i}}_{j} = 0\,, \\ &{{O_{(I)}}^{ij}}_{kl} = 0\,, {{O_{(II)}}^{ij}}_{kl} = 0\,, {{O_{(III)}}^{ij}}_{kl} = 0\,, O_{(IV)ijklmn} = 0\,, \\ &{{O_{(V)}}^{ijk}}_{lmn} = 0\,, {{O_{(VI)}}^{i}}_{j} = 0\,, \\ &{{P_{(I)}}^{ij}}_{kl} = 0\,, {{P_{(II)}}^{ij}}_{kl} = 0\,, {{P_{(III)}}^{ij}}_{kl} = 0\,, P_{(IV)ijklmn} = 0\,, \\ &{{P_{(V)}}^{ijk}}_{lmn} = 0\,, {{P_{(VI)}}^{ijk}}_{lmn} = 0\,, {{P_{(VII)}}^{ijk}}_{lmn} = 0\,, \\ &{{U_{(I)}}^{ij}}_{kl} = 0\,, U_{(II)ijklmn} = 0\,, {{U_{(III)}}^{ijk}}_{lmn} = 0\,, {{U_{(IV)}}^{ijk}}_{lmn} = 0\,, \\ &{U_{(V)}}^{ijklmn} = 0\,, {{U_{(VI)}}^{ijk}}_{lmn} = 0\,, \\ &{{V_{(I)}}^{ijk}}_{lmn} = 0\,, {V_{(II)}}^{ijklmn} = 0\,, {{V_{(III)}}^{ijk}}_{lmn} = 0\,, \\ &Z^{ijklmn} = 0 \quad \}\,. \end{aligned} \tag{5.263}$$

5.7 3 U(7) vortices on $\mathbb{C}$

The Hilbert series for the 3 U(7) vortex master space can be obtained by solving the following Molien integral,

$$g(t,s,x;\widetilde{\mathcal{F}^\flat}_{3,7}) = \oint \mathrm{d}\mu_{\mathrm{SU}(3)} \mathrm{PE}\Big[[0,1]_w[1,0,0,0,0,0]_x + [1,1]_w s\Big]\,, \tag{5.264}$$

where $[1,0,0,0,0,0]_w$ is the fundamental representation of the global SU(7). The integral leads to the following character expansion of the Hilbert series

$$\begin{aligned} g(t,s,x;\widetilde{\mathcal{F}^\flat}_{3,7}) = &\frac{1}{(1-s^2)(1-s^3)} \times \\ &\sum_{n_0=0}^{\infty}\sum_{n_1=0}^{\infty}\sum_{n_2=0}^{\infty}\sum_{n_3=0}^{\infty} \Big[[n_1+n_2+3n_3, n_1+n_2, n_0, 0,0,0]_x s^{n_1+2n_2+3n_3} t^{3n_0+3n_1+3n_2+3n_3} \\ &+ [n_1+n_2, n_1+n_2+3n_3+3, n_0, 0,0,0]_x s^{n_1+2n_2+3n_3+3} t^{3n_0+3n_1+3n_2+6n_3+6}\Big]\,, \end{aligned} \tag{5.265}$$

where $[m_1, m_2, m_3, m_4, m_5, m_6]_x$ is a character of a SU(7) irreducible representation with highest weights $m_1, \ldots, m_6$.

The plethystic logarithm of the Hilbert series is

$$
\begin{aligned}
\mathrm{PL}\Big[g(t,s,x;\widetilde{\mathcal{F}^\flat}_{3,7})\Big] =\ & s^2+s^3+[0,0,1,0,0,0]_x t^3+[1,1,0,0,0,0]_x s t^3 \\
& +[1,1,0,0,0,0]_x s^2 t^3+[3,0,0,0,0,0]_x s^3 t^3 \\
& -[1,0,0,0,1,0]_x t^6-([2,0,0,1,0,0]_x+[0,1,0,1,0,0]_x \\
& +[1,0,0,0,1,0]_x) s t^6-([1,1,1,0,0,0]_x+[0,0,2,0,0,0]_x \\
& +[2,0,0,1,0,0]_x+[0,1,0,1,0,0]_x+[1,0,0,0,1,0]_x) s^2 t^6 \\
& +([1,1,0,0,0,0]_x+[0,0,0,1,1,0]_x) t^9-([3,0,1,0,0,0]_x \\
& +2[1,1,1,0,0,0]_x+[0,0,2,0,0,0]_x+2[2,0,0,1,0,0]_x \\
& +[0,1,0,1,0,0,0]_x+[1,0,0,0,1,0,0]_x) s^3 t^6+\dots \\
& -([0,0,2,0,0,0]_x+[1,1,1,0,0,0]_x+[2,0,0,1,0,0]_x \\
& +[2,2,0,0,0,0]_x+[3,0,1,0,0,0]_x) s^4 t^6+\dots \\
& -([1,1,1,0,0]_x+[2,2,0,0,0]_x+[3,0,1,0,0]_x) s^5 t^6+\dots \\
& -[2,2,0,0,0]_x s^6 t^6+\dots\,.
\end{aligned} \tag{5.266}
$$

The generators of the vortex master space are indicated by the above plethystic logarithm. They are as follows,

$$
\begin{aligned}
s^2 &\to u_2=\mathrm{Tr}(\phi^2) \\
s^3 &\to u_3=\mathrm{Tr}(\phi^3) \\
[0,0,1,0,0,0]_x t^3 &\to B^{ijk}=\epsilon^{\alpha_1\alpha_2\alpha_3}Q^i_{\alpha_1}Q^j_{\alpha_2}Q^k_{\alpha_3} \\
[1,1,0,0,0,0]_x s t^3 &\to \begin{cases} {A_{001}}^{ijk}=\epsilon^{\alpha_1\alpha_2\alpha_3}Q^i_{\alpha_1}Q^j_{\alpha_2}\phi^{\beta}_{\alpha_3}Q^k_{\beta} \\ \epsilon_{ijkmnpo}{A_{001}}^{ijk}=0 \end{cases} \\
[1,1,0,0,0,0]_x s^2 t^3 &\to \begin{cases} {A_{002}}^{ijk}=\epsilon^{\alpha_1\alpha_2\alpha_3}Q^i_{\alpha_1}Q^j_{\alpha_2}\phi^{\beta_1}_{\alpha_3}\phi^{\beta_2}_{\beta_1}Q^k_{\beta_2} \\ \epsilon_{ijkmnpo}{A_{002}}^{ijk}=-\frac{1}{3}u_2\epsilon_{mnporsu}B^{rsu} \\ {A_{011}}^{ijk}=\epsilon^{\alpha_1\alpha_2\alpha_3}Q^k_{\alpha_1}\phi^{\beta_1}_{\alpha_2}Q^i_{\beta_1}\phi^{\beta_2}_{\alpha_3}Q^j_{\beta_2} \\ \epsilon_{ijkmnpo}{A_{011}}^{ijk}=\frac{1}{6}u_2\epsilon_{mnporsu}B^{rsu} \\ \to {A_{002}}^{ijk}={A_{011}}^{ijk}+\frac{1}{2}u_2B^{ijk} \end{cases} \\
[3,0,0,0,0,0]_x s^3 t^3 &\to \begin{cases} {A_{012}}^{ijk}=\epsilon^{\alpha_1\alpha_2\alpha_3}Q^i_{\alpha_1}\phi^{\beta_1}_{\alpha_2}Q^j_{\beta_1}\phi^{\beta_2}_{\alpha_3}\phi^{\beta_3}_{\beta_2}Q^k_{\beta_3} \\ {S_{012}}^{ijk}={A_{012}}^{ijk}+{A_{012}}^{jki}+{A_{012}}^{kij} \end{cases} .
\end{aligned}
$$

Quadratic relations. The plethystic logarithm indicates the quadratic relations formed amongst the generators,

$$
\begin{gathered}
-[1,0,0,0,1,0]_x t^6 \\
-[0,1,0,1,0,0]_x s t^6 - [2,0,0,1,0,0]_x s t^6 - [1,0,0,0,1,0]_x s t^6 \\
-[0,1,0,1,0,0]_x s^2 t^6 - 2[2,0,0,1,0,0]_x s^2 t^6 - [0,0,2,0,0,0]_x s^2 t^6 \\
-[1,1,1,0,0,0]_x s^2 t^6 - [1,0,0,0,1,0]_x s^2 t^6 \\
-[0,1,0,1,0,0]_x s^3 t^6 - 2[2,0,0,1,0,0]_x s^3 t^6 - [0,0,2,0,0,0]_x s^3 t^6 \\
-2[1,1,1,0,0,0]_x s^3 t^6 - [3,0,1,0,0,0]_x s^3 t^6 \\
-[2,0,0,1,0,0]_x s^4 t^6 - [0,0,2,0,0,0]_x s^4 t^6 - 2[1,1,1,0,0,0]_x s^4 t^6 \\
-[2,2,0,0,0,0]_x s^4 t^6 - [3,0,1,0,0,0]_x s^4 t^6 \\
-[1,1,1,0,0,0]_x s^5 t^6 - [2,2,0,0,0,0]_x s^5 t^6 - [3,0,1,0,0,0]_x s^5 t^6 \\
-[2,2,0,0,0,0]_x s^6 t^6 \,.
\end{gathered} \tag{5.267}
$$

The quadratic relations are formed by the generators of the vortex master space which are:

$$u_2\,, u_3\,, B^{ijk}\,, {A_{001}}^{ijk}\,, {A_{002}}^{ijk}\,, {S_{012}}^{ijk}\,. \tag{5.268}$$

For the first quadratic relation at order t^6 we consider the following SU(7) representation product,

$$\mathrm{Sym}^2[0,0,1,0,0,0]_x = [0,0,2,0,0,0]_x + [1,0,0,0,1,0]_x\,. \tag{5.269}$$

The above symmetric product allows us to construct the following quadratic relation:

- $-[1,0,0,0,1,0]_x t^6$ *relations.* We consider the following generator product,

$$H_{ij}{}^{k} = \epsilon_{ijpqrsu} B^{pqr} B^{suk} = 0\,, \tag{5.270}$$

  which vanishes exactly. It is exactly the quadratic relation at this order.

For the second set of quadratic relations containing the order st^6, we consider the following representation products,

$$
\begin{aligned}
[1,1,0,0,0,0]_x \times [0,0,1,0,0,0]_x = &\; [1,1,1,0,0,0]_x + [2,0,0,1,0,0]_x \\
&+ [0,1,0,1,0,0]_x + [1,0,0,0,1,0]_x\,.
\end{aligned} \tag{5.271}
$$

The above tensor product guides us in constructing the following quadratic relations:

- $-[0,1,0,1,0,0]_x st^6$ *relations.* We consider the following generator product for this order,

$$R_{(I)}{}^{ij}{}_{klo} = {A_{001}}^{ijm} B^{pqr} \epsilon_{mpqrklo} = 0\,, \tag{5.272}$$

  which vanishes exactly. This transforms in the correct representation of SU(7) and is the quadratic relation at this order.

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- $-[2,0,0,1,0,0]_x st^6$ *relations.* The following generator product vanishes exactly,

$$R_{(II)}{}^{ij}{}_{klo} = A_{001}{}^{imj} B^{pqr} \epsilon_{mpqrklo} = 0\,. \tag{5.273}$$

  It is in the correct representation and hence is the quadratic relation at this order.

- $-[1,0,0,0,1,0]_x st^6$ *relations.* For this order, the following generator product is considered,

$$R_{(III)}{}^{i}{}_{jo} = A_{001}{}^{imn} B^{klp} \epsilon_{mnklpjo} = 0\,. \tag{5.274}$$

  The above product exactly vanishes. It is the quadratic relation at this order.

The next set of quadratic relations contains the order s^2t^6. We consider the following SU(7) representation products in order to construct the relations,

$$\begin{aligned}
\mathrm{Sym}^2[1,1,0,0,0,0]_x &= [2,2,0,0,0,0]_x + [1,1,1,0,0,0]_x + [2,0,0,1,0,0]_x \\
&\quad + [0,0,2,0,0,0]_x\,, \\
[1,1,0,0,0,0]_x \times [0,0,1,0,0,0]_x &= [1,1,1,0,0,0]_x + [2,0,0,1,0,0]_x + [0,1,0,1,0,0]_x \\
&\quad + [1,0,0,0,1,0]_x\,, \\
\mathrm{Sym}^2[0,0,1,0,0,0]_x &= [0,0,2,0,0,0]_x + [1,0,0,0,1,0]_x\,.
\end{aligned} \tag{5.275}$$

The above representation products guide us in constructing the following quadratic relations:

- $-[0,1,0,1,0,0]_x s^2t^6$ *relations.* We consider the following generator product for this order,

$$O_{(I)}{}^{ij}{}_{klo} = A_{002}{}^{ijp} B^{qrs} \epsilon_{pqrsklo} = 0\,, \tag{5.276}$$

  which exactly vanishes. The above is precisely the quadratic relation for this order.

- $-2[2,0,0,1,0,0]_x s^2t^6$ *relations.* We consider the following generator products for the quadratic relation at this order,

$$\begin{aligned}
O_{(II)}{}^{ij}{}_{klo} &= A_{001}{}^{mni} A_{001}{}^{pqj} \epsilon_{mnpqklo} = 0\,, \\
O_{(III)}{}^{ij}{}_{klo} &= A_{002}{}^{imj} B^{npq} \epsilon_{mnpqklo} = 0\,,
\end{aligned} \tag{5.277}$$

  where the products above both vanish exactly. The above are the two distinct quadratic relations at this order.

- $-[0,0,2,0,0,0]_x s^2t^6$ *relations.* The following generator products are helpful in constructing the quadratic relations at this order,

$$\begin{aligned}
o_{(I)}{}_{ijklmno_1o_2} &= \epsilon_{pqsijko_1} \epsilon_{uvrlmno_2} A_{001}{}^{pqr} A_{001}^{uvs}\,, \\
o_{(II)}{}_{ijklmno_1o_2} &= u_2 B^{pqs} B^{uvr} \epsilon_{pqsijko_1} \epsilon_{uvrlmno_2}\,.
\end{aligned} \tag{5.278}$$

  The above products satisfy the following quadratic relation,

$$O_{(IV)}{}_{ijklmno_1o_2} = o_{(I)}{}_{ijklmno_1o_2} - \frac{1}{9} o_{(II)}{}_{ijklmno_1o_2}\,, \tag{5.279}$$

  which is precisely the relation at this order.

- $-[1,1,1,0,0,0]_x s^2t^6$ *relations.* We consider the following generator products for the relation at this order,

$$
\begin{aligned}
{o_{(III)}}^{ijk}{}_{lmno} &= {A_{001}}^{pqk}{A_{001}}^{ijr}\epsilon_{pqrlmno}\,,\\
{o_{(IV)}}^{ijk}{}_{lmno} &= {A_{002}}^{ijk}B^{pqr}\epsilon_{pqrlmno}\,,
\end{aligned}
\tag{5.280}
$$

which transform in the correct representation at this order. The above products satisfy the following quadratic relation,

$$
{O_{(V)}}^{ijk}{}_{lmno} = {o_{(III)}}^{ijk}{}_{lmno} - \frac{1}{3}{o_{(IV)}}^{ijk}{}_{lmno} = 0\,,
\tag{5.281}
$$

which is precisely the relation at this order.

- $-[1,0,0,0,1,0]_x s^2t^6$ *relations.* The following generator product vanishes exactly,

$$
{O_{(VI)}}^{i}{}_{jo} = {A_{002}}^{pqi}B^{lmn}\epsilon_{pqlmnjo} = 0\,.
\tag{5.282}
$$

The above is the quadratic relations for this order.

The next set of quadratic relations are at orders of s^3t^6. We first consider the following SU(7) representation products,

$$
\begin{aligned}
[1,1,0,0,0,0]_x\times[1,1,0,0,0,0]_x &= [2,2,0,0,0,0]_x+[3,0,1,0,0,0]_x\\
&\quad +[0,3,0,0,0,0]_x+2[1,1,1,0,0,0]_x\\
&\quad +[2,0,0,1,0,0]_x+[0,0,2,0,0,0]_x\\
&\quad +[0,1,0,1,0,0]_x\,,\\
[3,0,0,0,0,0]_x\times[0,0,1,0,0,0]_x &= [3,0,1,0,0,0]_x+[2,0,0,1,0,0]_x\,,\\
[1,1,0,0,0,0]_x\times[0,0,1,0,0,0]_x &= [1,1,1,0,0,0]_x+[2,0,0,1,0,0]_x\\
&\quad +[0,1,0,1,0,0]_x+[1,0,0,0,1,0]_x\,,\\
\mathrm{Sym}^2[0,0,1,0,0,0]_x &= [0,0,2,0,0,0]_x+[1,0,0,0,1,0]_x\,.
\end{aligned}
\tag{5.283}
$$

The above representation products lead us to the following quadratic relations:

- $-[0,1,0,1,0,0]_x s^3t^6$ *relations.* The following generator product vanishes exactly,

$$
{P_{(I)}}^{ij}{}_{klo} = {A_{001}}^{ijm}{A_{002}}^{pqr}\epsilon_{mpqrklo} = 0\,,
\tag{5.284}
$$

and transforms in the representation of this order. Accordingly, it is exactly the quadratic relation we are looking for this order.

- $-2[2,0,0,1,0,0]_x s^3t^6$ *relations.* For this order, we consider the following generator products,

$$
\begin{aligned}
{P_{(II)}}^{ij}{}_{klo} &= {A_{001}}^{mni}{A_{002}}^{pqj}\epsilon_{mnpqklo} = 0\,,\\
{P_{(III)}}^{ij}{}_{klo} &= {S_{012}}^{ijm}B^{npq}\epsilon_{mnpqklo} = 0\,.
\end{aligned}
\tag{5.285}
$$

Both above vanish and satisfy the correct transformation property for this order. They are precisely the two quadratic relations at this order.

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- $-[0,0,2,0,0,0]_x s^3t^6$ *relations.* We first consider the following generator products,

$$\begin{aligned} p_{(I)}{}_{ijklmno_1o_2} &= A_{001}{}^{pqu}A_{002}{}^{rsv}\epsilon_{pqvijko_1}\epsilon_{rsulmno_2}\,, \\ p_{(II)}{}_{ijklmno_1o_2} &= u_3 B^{pqu}B^{rsv}\epsilon_{pquijko_1}\epsilon_{rsvlmno_2}\,, \end{aligned} \tag{5.286}$$

which transform in the correct representation corresponding to this order. The products satisfy the following quadratic relation

$$P_{(IV)}{}_{ijklmno_1o_2} = p_{(I)}{}_{ijklmno_1o_2} - \frac{1}{9}p_{(II)}{}_{ijklmno_1o_2}\,, \tag{5.287}$$

which is the relation for this order.

- $-2[1,1,1,0,0,0]_x s^3t^6$ *relations.* For the quadratic relation at this order, we need to consider the following generator products,

$$\begin{aligned} p_{(III)}{}^{ijk}{}_{lmno} &= A_{001}{}^{ijp}A_{002}{}^{qrk}\epsilon_{pqrlmno}\,, \\ p_{(IV)}{}^{ijk}{}_{lmno} &= A_{002}{}^{ijp}A_{001}{}^{qrk}\epsilon_{pqrlmno}\,, \\ p_{(V)}{}^{ijk}{}_{lmno} &= u_2 A_{001}{}^{ijp}B^{qrk}\epsilon_{pqrlmno}\,. \end{aligned} \tag{5.288}$$

The above products transform in the correct representation for this order. The quadratic relations formed by the above are

$$\begin{aligned} P_{(V)}{}^{ijk}{}_{lmno} &= p_{(III)}{}^{ijk}{}_{lmno} - p_{(IV)}{}^{ijk}{}_{lmno} = 0\,, \\ P_{(VI)}{}^{ijk}{}_{lmno} &= p_{(III)}{}^{ijk}{}_{lmno} - \frac{1}{2}p_{(V)}{}^{ijk}{}_{lmno} = 0\,, \end{aligned} \tag{5.289}$$

exactly corresponding to the two expected quadratic relations at this order.

- $-[3,0,1,0,0,0]_x s^3t^6$ *relations.* We consider the following generator products for the quadratic relation at this order,

$$\begin{aligned} p_{(VI)}{}^{ijk}{}_{lmno} &= (A_{001}{}^{ipj}A_{002}{}^{qrk} + A_{001}{}^{jpk}A_{002}{}^{qri} + A_{001}{}^{kpi}A_{002}{}^{qrj})\epsilon_{pqrlmno}\,, \\ p_{(VII)}{}^{ijk}{}_{lmno} &= S_{012}{}^{ijk}B^{pqr}\epsilon_{pqrlmno}\,, \end{aligned} \tag{5.290}$$

which transform in the correct representation for this order. The above products satisfy the following quadratic relation

$$P_{(VII)}{}^{ijk}{}_{lmno} = p_{(VI)}{}^{ijk}{}_{lmno} + \frac{1}{3}p_{(VII)}{}^{ijk}{}_{lmno}\,, \tag{5.291}$$

which is precisely the relation for this order.

The next set of quadratic relations are of orders s^4t^6. In order to construct the relations, we consider the following SU(7) representation products,

$$\begin{aligned} \mathrm{Sym}^2[1,1,0,0,0,0]_x &= [2,2,0,0,0,0]_x + [1,1,1,0,0,0]_x \\ &\quad + [2,0,0,1,0,0]_x + [0,0,2,0,0,0]_x\,, \\ [1,1,0,0,0,0]_x \times [3,0,0,0,0,0]_x &= [4,1,0,0,0,0]_x + [2,2,0,0,0,0]_x \\ &\quad + [3,0,1,0,0,0]_x + [1,1,1,0,0,0]_x\,, \\ [1,1,0,0,0,0]_x \times [0,0,1,0,0,0]_x &= [1,1,1,0,0,0]_x + [2,0,0,1,0,0]_x \\ &\quad + [0,1,0,1,0,0]_x + [1,0,0,0,1,0]_x\,, \\ \mathrm{Sym}^2[0,0,1,0,0,0]_x &= [0,0,2,0,0,0]_x + [1,0,0,0,1,0]_x\,. \end{aligned} \tag{5.292}$$

The above representation products help us in constructing the quadratic relations as follows:

- $-[2,0,0,1,0,0]_x s^4 t^6$ *relations.* We consider the following generator product for the quadratic relation at this order,

$$U_{(I)}{}^{ij}{}_{klo} = A_{002}{}^{pqi} A_{002}{}^{mnj} \epsilon_{pqmnklo} = 0\,, \tag{5.293}$$

which vanishes exactly. This is precisely the quadratic relation at this order.

- $-[0,0,2,0,0,0]_x s^4 t^6$ *relations.* For this order, we consider the following generator products for the quadratic relation,

$$\begin{aligned} u_{(I)}{}_{ijklmno_1o_2} &= A_{002}{}^{pqr} A_{002}{}^{uvs} \epsilon_{pqsijko_1} \epsilon_{uvrlmno_2}\,, \\ u_{(II)}{}_{ijklmno_1o_2} &= u_2 u_2 B^{pqr} B^{uvw} \epsilon_{pqrijko_1} \epsilon_{uvwlmno_2}\,. \end{aligned} \tag{5.294}$$

The above products transform in the correct SU(7) representation of this order. They satisfy the following quadratic relation,

$$U_{(II)}{}_{ijklmno_1o_2} = u_{(I)}{}_{ijklmno_1o_2} - \frac{1}{3} u_{(II)}{}_{ijklmno_1o_2} = 0\,, \tag{5.295}$$

which is precisely the relation we are looking for here.

- $-2[1,1,1,0,0,0]_x s^4 t^6$ *relations.* There are two distinct quadratic relations at this order. In order to construct them, we consider the following generator products,

$$\begin{aligned} u_{(III)}{}^{ijk}{}_{lmno} &= A_{002}{}^{ijp} A_{002}{}^{qrk} \epsilon_{pqrlmno}\,, \\ u_{(IV)}{}^{ijk}{}_{lmno} &= A_{001}{}^{ijp} S_{012}{}^{qrk} \epsilon_{pqrlmno}\,, \\ u_{(V)}{}^{ijk}{}_{lmno} &= u_2 A_{001}{}^{ijk} A_{001}{}^{pqr} \epsilon_{pqrlmno}\,, \\ u_{(VI)}{}^{ijk}{}_{lmno} &= u_3 A_{001}{}^{ijk} B^{pqr} \epsilon_{pqrlmno}\,, \end{aligned} \tag{5.296}$$

which transform in the representation of this order. The above products form the following two quadratic relations,

$$\begin{aligned} U_{(III)}{}^{ijk}{}_{lmno} &= u_{(V)}{}^{ijk}{}_{lmno} - 2u_{(III)}{}^{ijk}{}_{lmno} + \frac{2}{9} u_{(VI)}{}^{ijk}{}_{lmno} = 0\,, \\ U_{(IV)}{}^{ijk}{}_{lmno} &= u_{(IV)}{}^{ijk}{}_{lmno} + \frac{1}{6} u_{(VI)}{}^{ijk}{}_{lmno} = 0\,. \end{aligned} \tag{5.297}$$

The above are the two quadratic relations at this order.

- $-[2,2,0,0,0,0]_x s^4 t^6$ *relations.* The following generator product with its symmetrization and anti-symmetrization of indices is required for the construction of the quadratic relation at this order,

$$u_{(VII)}{}^{ijklmn} = A_{001}{}^{ijk} S_{012}{}^{lmn}\,, \tag{5.298}$$

where we antisymmetrize on the indices $[kl]$ and $[mn]$ as follows,

$$u_{(VIII)}{}^{ijklmn} = u_{(VII)}{}^{ijklmn} - u_{(VII)}{}^{ijlkmn} - u_{(VII)}{}^{ijklnm} + u_{(VII)}{}^{ijlknm}\,. \tag{5.299}$$

A further symmetrization on the two paired indices $[kl]$ and $[mn]$ leads to the following

$$U_{(V)}{}^{ijklmn} = u_{(VIII)}{}^{ijklmn} + u_{(VIII)}{}^{ijmnkl} = 0\,, \tag{5.300}$$

which exactly vanishes. This is precisely the quadratic relation at this order we are looking for.

- $-[3,0,1,0,0,0]_x s^4 t^6$ *relations.* The quadratic relation at this order is formed by

$$U_{(VI)}{}^{ijk}{}_{lmno} = (A_{001}{}^{ipq} S_{012}{}^{rjk} + A_{001}{}^{jpq} S_{012}{}^{rki} + A_{001}{}^{jpq} S_{012}{}^{rij})\epsilon_{pqrlmno} = 0\,, \tag{5.301}$$

where the above contains the symmetrization of the generator product

$$A_{001}{}^{ipq} S_{012}{}^{rjk} \epsilon_{pqrlmno} \tag{5.302}$$

in the indices ijk. The above quadratic relation satisfies precisely the transformation properties for this order and is the relation we are looking for.

For the next set of quadratic relations at orders of $s^5 t^6$, we consider the following representation products in order to construct the relations,

$$\begin{aligned}
[3,0,0,0,0,0]_x \times [1,1,0,0,0,0]_x &= [4,1,0,0,0,0]_x + [2,2,0,0,0,0]_x \\
&\quad + [3,0,1,0,0,0]_x + [1,1,1,0,0,0]_x\,, \\
[1,1,0,0,0,0]_x \times [1,1,0,0,0,0]_x &= [2,2,0,0,0,0]_x + [3,0,1,0,0,0]_x \\
&\quad + [0,3,0,0,0,0]_x + 2[1,1,1,0,0,0]_x \\
&\quad + [2,0,0,1,0,0]_x + [0,0,2,0,0,0]_x \\
&\quad + [0,1,0,1,0,0]_x\,, \\
[3,0,0,0,0,0]_x \times [0,0,1,0,0,0]_x &= [3,0,1,0,0,0]_x + [2,0,0,1,0,0]_x\,, \\
[1,1,0,0,0,0]_x \times [0,0,1,0,0,0]_x &= [1,1,1,0,0,0]_x + [2,0,0,1,0,0]_x \\
&\quad + [0,1,0,1,0,0]_x + [1,0,0,0,1,0]_x\,, \\
\mathrm{Sym}^2[1,1,0,0,0,0]_x &= [2,2,0,0,0,0]_x + [1,1,1,0,0,0]_x \\
&\quad + [2,0,0,1,0,0]_x + [0,0,2,0,0,0]_x\,, \\
\mathrm{Sym}^2[0,0,1,0,0,0]_x &= [0,0,2,0,0,0]_x + [1,0,0,0,1,0]_x\,.
\end{aligned} \tag{5.303}$$

The above products are used to construct the following quadratic relations of generators:

- $-[1,1,1,0,0,0]_x s^5 t^6$ *relations.* We consider the following products of generators for the quadratic relation at this order,

$$\begin{aligned}
v_{(I)}{}^{ijk}{}_{lmno} &= A_{002}{}^{ijp} S_{012}{}^{qrk} \epsilon_{pqrlmno}\,, \\
v_{(II)}{}^{ijk}{}_{lmno} &= u_3 A_{001}{}^{ijp} A_{001}{}^{qrk} \epsilon_{pqrlmno}\,,
\end{aligned} \tag{5.304}$$

which transform in the correct irreducible representation of this order. The above products satisfy the following quadratic relation,

$$V_{(I)}{}^{ijk}{}_{lmno} = v_{(I)}{}^{ijk}{}_{lmno} + \frac{1}{2} v_{(II)}{}^{ijk}{}_{lmno} = 0 \,. \tag{5.305}$$

The above precisely is the quadratic relation we are looking for at this order.

- $-[2,2,0,0,0,0]_x s^5 t^6$ *relations.* We need to consider several generator products with various symmetrizations and anti-symmetrizations of indices in order to construct the quadratic relation for this order. The first product to consider is the following,

$$v_{(III)}{}^{ijklmn} = \left(A_{002}{}^{ijk} - \frac{1}{2} u_2 B^{ijk} \right) S_{012}{}^{lmn} \,, \tag{5.306}$$

where we recall that $A_{011}{}^{ijk} = A_{002}{}^{ijk} - \frac{1}{2} u_2 B^{ijk}$. Above, we anti-symmetrize in the indices $[kl]$ and $[mn]$ to give

$$v_{(IV)}{}^{ijklmn} = v_{(III)}{}^{ijklmn} - v_{(III)}{}^{ijlkmn} - v_{(III)}{}^{ijklnm} + v_{(III)}{}^{ijlknm} \,, \tag{5.307}$$

and further anti-symmetrize in the pairs of indices $[ij]$ and $[mn]$ to obtain,

$$v_{(V)}{}^{ijklmn} = v_{(IV)}{}^{ijklmn} - v_{(IV)}{}^{mnklij} \,. \tag{5.308}$$

The second generator product to consider is the following,

$$v_{(VI)}{}^{ijklmn} = u_3 A_{001}{}^{ijk} A_{001}{}^{lmn} \,, \tag{5.309}$$

which we anti-symmetrize in the indices $[kl]$ and $[mn]$ giving,

$$v_{(VII)}{}^{ijklmn} = v_{(VI)}{}^{ijklmn} - v_{(VI)}{}^{ijlkmn} - v_{(VI)}{}^{ijklnm} + v_{(VI)}{}^{ijlknm} \,. \tag{5.310}$$

A further symmetrization of the pair of indices $[ij]$ and $[kl]$ gives

$$v_{(VIII)}{}^{ijklmn} = v_{(VII)}{}^{ijklmn} + v_{(VII)}{}^{klijmn} \,. \tag{5.311}$$

From the above generator products, which correctly transform at this order, we can construct the following quadratic relation,

$$V_{(II)}{}^{ijklmn} = v_{(V)}{}^{ijklmn} + v_{(VIII)}{}^{ijmnkl} = 0 \,. \tag{5.312}$$

This is precisely we are looking for at this order.

- $-[3,0,1,0,0,0]_x s^5 t^6$ *relations.* For the quadratic relation at this order, we need to consider the following generators products,

$$v_{(IX)}{}^{ijk}{}_{lmno} = A_{002}{}^{ipq} S_{012}{}^{rjk} \epsilon_{pqrlmno} \,, \tag{5.313}$$

which we symmetrize in the indices ijk as follows,

$$v_{(X)}{}^{ijk}{}_{lmno} = v_{(IX)}{}^{ijk}{}_{lmno} + v_{(IX)}{}^{jki}{}_{lmno} + v_{(IX)}{}^{kij}{}_{lmno} \,. \tag{5.314}$$

The second generator product which we need to consider is the following,

$$v_{(XI)}{}^{ijk}{}_{lmno} = u_2 A_{001}{}^{ipj} A_{002}{}^{qrk} \epsilon_{pqrlmno} \,, \tag{5.315}$$

which we symmetrize in the indices ijk as follows,

$$v_{(XII)}{}^{ijk}{}_{lmno} = v_{(XI)}{}^{ijk}{}_{lmno} + v_{(XI)}{}^{jki}{}_{lmno} + v_{(XI)}{}^{kij}{}_{lmno} \,. \tag{5.316}$$

From the above products, we construct the following quadratic relation,

$$V_{(III)}{}^{ijk}{}_{lmno} = v_{(X)}{}^{ijk}{}_{lmno} - \frac{1}{3} v_{(XII)}{}^{ijk}{}_{lmno} = 0 \,. \tag{5.317}$$

This is precisely the relation we are looking for at this order of the plethystic logarithm.

The final quadratic relation is of order s^6t^6. It can be constructed by considering the following SU(7) representation products

$$\begin{aligned}
\mathrm{Sym}^2[3,0,0,0,0,0]_x &= [6,0,0,0,0,0]_x + [2,2,0,0,0,0]_x \,, \\
\mathrm{Sym}^2[1,1,0,0,0,0]_x &= [2,2,0,0,0,0]_x + [1,1,1,0,0,0]_x \\
&\quad + [2,0,0,1,0,0]_x + [0,0,2,0,0,0]_x \,, \\
[1,1,0,0,0,0]_x \times [3,0,0,0,0,0]_x &= [4,1,0,0,0,0]_x + [2,2,0,0,0,0]_x \\
&\quad + [3,0,1,0,0,0]_x + [1,1,1,0,0,0]_x \,, \\
[1,1,0,0,0,0]_x \times [1,1,0,0,0,0]_x &= [2,2,0,0,0,0]_x + [3,0,1,0,0,0]_x \\
&\quad + [0,3,0,0,0,0]_x + 2[1,1,1,0,0,0]_x \\
&\quad + [2,0,0,1,0,0]_x + [0,0,2,0,0,0]_x \\
&\quad + [0,1,0,1,0,0]_x \,.
\end{aligned} \tag{5.318}$$

From the above, we can construct candidate products of generators of the vortex master space, in order to identify quadratic relations amongst them:

- $-[2,2,0,0,0,0]_x s^6 t^6$ *relations.* The final quadratic relation can be identified from the following generator product and its symmetrization and anti-symmetrization of indices,

$$z_{(I)}{}^{ijklmn} = S_{012}{}^{ijk} S_{012}{}^{lmn} \,. \tag{5.319}$$

We anti-symmetrize the above on the indices $[kl]$ and $[mn]$

$$z_{(II)}{}^{ijklmn} = z_{(I)}{}^{ijklmn} - z_{(I)}{}^{ijlkmn} - z_{(I)}{}^{ijklnm} + z_{(I)}{}^{ijlknm} \,, \tag{5.320}$$

and further symmetrize on the pairs of indices $[kl]$ and $[mn]$ to give

$$Z^{ijklmn} = z_{(II)}{}^{ijklmn} + z_{(II)}{}^{ijmnkl} = 0 \,. \tag{5.321}$$

The above quadratic relation vanishes non-trivially and by construction transforms in the representation of this order of the plethystic logarithm. The above is the quadratic relation corresponding to this order.

Vortex moduli space. Given the above explicit computation of the quadratic relations between generators, we can proceed in identifying the vortex moduli space for the 3 U(7) vortex theory. The vortex moduli space can be expressed as a $\mathbb{C}^*$ projection as follows,

$$\begin{aligned}\widetilde{\mathcal{V}}_{3,7} = \widetilde{\mathcal{F}^\flat}_{3,7}/\{B^{ijk} \simeq \lambda^3 B^{ijk}, {A_{001}}^{ijk} \simeq \lambda^3 {A_{001}}^{ijk},\\ {A_{002}}^{ijk} \simeq \lambda^3 {A_{002}}^{ijk}, {S_{012}}^{ijk} \simeq \lambda^3 {S_{012}}^{ijk}\}\,.\end{aligned} \tag{5.322}$$

The master space of the vortex theory can be identified with the help of the quadratic relations as follows,

$$\begin{aligned}\widetilde{\mathcal{F}^\flat}_{3,7} = \mathbb{C}[u_2, u_3, B^{ijk}, {A_{001}}^{ijk}, {A_{002}}^{ijk}, {S_{012}}^{ijk}]/\{&\\ {H_{ij}}^{k} = 0\,,&\\ {R_{(I)}}^{ij}{}_{klo} = 0\,, {R_{(II)}}^{ij}{}_{klo} = 0\,, {R_{(III)}}^{i}{}_{jo} = 0\,,&\\ {O_{(I)}}^{ij}{}_{klo} = 0\,, {O_{(II)}}^{ij}{}_{klo} = 0\,, {O_{(III)}}^{ij}{}_{klo} = 0\,, O_{(IV)ijklmno_1o_2} = 0\,,&\\ {O_{(V)}}^{ijk}{}_{lmno} = 0\,, {O_{(VI)}}^{i}{}_{jo} = 0\,,&\\ {P_{(I)}}^{ij}{}_{klo} = 0\,, {P_{(II)}}^{ij}{}_{klo} = 0\,, {P_{(III)}}^{ij}{}_{klo} = 0\,, P_{(IV)ijklmno_1o_2} = 0\,,&\\ {P_{(V)}}^{ijk}{}_{lmno} = 0\,, {P_{(VI)}}^{ijk}{}_{lmno} = 0\,, {P_{(VII)}}^{ijk}{}_{lmno} = 0\,,&\\ {U_{(I)}}^{ij}{}_{klo} = 0\,, U_{(II)ijklmno_1o_2} = 0\,, {U_{(III)}}^{ijk}{}_{lmno} = 0\,, {U_{(IV)}}^{ijk}{}_{lmno} = 0\,,&\\ {U_{(V)}}^{ijklmn} = 0\,, {U_{(VI)}}^{ijk}{}_{lmno} = 0\,,&\\ {V_{(I)}}^{ijk}{}_{lmno} = 0\,, {V_{(II)}}^{ijklmn} = 0\,, {V_{(III)}}^{ijk}{}_{lmno} = 0\,,&\\ Z^{ijklmn} = 0 \quad \}\,.&\end{aligned} \tag{5.323}$$

5.8 3 U(N) vortices on $\mathbb{C}$

In this section, we summarize the generalization of the 3 vortex moduli space beyond U(7). We begin with the Hilbert series of the master space for 3 U(N) vortices. It can be computed using the following Molien integral

$$g(t,s,x;\widetilde{\mathcal{F}^\flat}_{3,N}) = \oint \mathrm{d}\mu_{\mathrm{SU}(3)} \mathrm{PE}\Big[[0,1]_w[1,0,\dots,0]_x t + [1,1]_w s\Big]\,. \tag{5.324}$$

The Hilbert series for the first few values of N are as follows

$$\begin{aligned} g(t,t;\widetilde{\mathcal{F}^\flat}_{3,1}) &= \frac{1}{(1-t^2)(1-t^3)(1-t^6)}\,,\\ g(t,t;\widetilde{\mathcal{F}^\flat}_{3,2}) &= \frac{1+2t^5+2t^6+t^{11}}{(1-t^2)(1-t^3)(1-t^4)^2(1-t^6)^2}\,,\\ g(t,t;\widetilde{\mathcal{F}^\flat}_{3,3}) &= \frac{1}{(1-t^2)(1-t^3)(1-t^4)^4(1-t^6)^3} \times (1+t^3+4t^4+8t^5+8t^6\\ &\quad +4t^7+9t^8+13t^9+14t^{10}+20t^{11}+14t^{12}+13t^{13}+9t^{14}+4t^{15}+8t^{16}\\ &\quad +8t^{17}+4t^{18}+t^{19}+t^{22})\,,\end{aligned}$$

$$
\begin{aligned}
g(t,t;\widetilde{\mathcal{F}^\flat}_{3,4}) = & \frac{1}{(1-t^2)(1-t^3)^5(1-t^4)^6(1-t^6)^4} \times (1+14t^4+20t^5 \\
& +16t^6-16t^7+5t^8+46t^9+94t^{10}+16t^{11}-94t^{12}-156t^{13}-135t^{14} \\
& -35t^{15}-t^{16}+65t^{17}-59t^{18}-161t^{19}-185t^{20}+125t^{21}+440t^{22} \\
& +440t^{23}+125t^{24}-185t^{25}-161t^{26}-59t^{27}+65t^{28}-t^{29}-35t^{30} \\
& -135t^{31}-156t^{32}-94t^{33}+16t^{34}+94t^{35}+46t^{36}+5t^{37}-16t^{38} \\
& +16t^{39}+20t^{40}+14t^{41}+t^{45}), \\
g(t,t;\widetilde{\mathcal{F}^\flat}_{3,5}) = & \frac{1}{(1-t^2)(1-t^3)^2(1-t^4)^8(1-t^6)^5} \times (1-t+9t^3+23t^4 \\
& +8t^5+30t^6+98t^7+200t^8+232t^9+320t^{10}+482t^{11}+677t^{12} \\
& +806t^{13}+800t^{14}+1052t^{15}+988t^{16}+485t^{17}+18t^{18}-127t^{19} \\
& -440t^{20}-970t^{21}-1728t^{22}-2074t^{23}-1778t^{24}-2074t^{25}-1728t^{26} \\
& -970t^{27}-440t^{28}-127t^{29}+18t^{30}+485t^{31}+988t^{32}+1052t^{33} \\
& +800t^{34}+806t^{35}+677t^{36}+482t^{37}+320t^{38}+232t^{39}+200t^{40} \\
& +98t^{41}+30t^{42}+8t^{43}+23t^{44}+9t^{45}-t^{47}+t^{48}), \qquad (5.325)
\end{aligned}
$$

where for simplicity we have set all global SU(N) fugacities to $x_i = 1$ and have set the adjoint and fundamental fugacities to be the same $s = t$.

As a character expansion, the Hilbert series for the 3 U(N) vortex master space is

$$
\begin{aligned}
& g(t,s,x;\widetilde{\mathcal{F}^\flat}_{3,N}) = \frac{1}{(1-s^2)(1-s^3)} \times \qquad (5.326) \\
& \sum_{n_0=0}^{\infty}\sum_{n_1=0}^{\infty}\sum_{n_2=0}^{\infty}\sum_{n_3=0}^{\infty} \Big[[n_1+n_2+3n_3, n_1+n_2, n_0, 0, \dots, 0]_x s^{n_1+2n_2+3n_3} t^{3n_0+3n_1+3n_2+3n_3} \\
& \quad + [n_1+n_2, n_1+n_2+3n_3+3, n_0, 0, \dots, 0]_x s^{n_1+2n_2+3n_3+3} t^{3n_0+3n_1+3n_2+6n_3+6} \Big],
\end{aligned}
$$

where $[m_1, \dots, m_{N-1}]_x$ represents the character of an irreducible representation of the global SU(N).

The plethystic logarithm takes the following form,

$$
\begin{aligned}
\mathrm{PL}\Big[g(t,s,x;\mathcal{F}^\flat_{3,N})\Big] = & \, s^2+s^3+[0,0,1,0,\dots,0]_x t^3+[1,1,0,0,0,0]_x st^3 \\
& +[1,1,0,0,\dots,0]_x s^2t^3+[3,0,0,0,\dots,0]_x s^3t^3 \\
& -[1,0,0,0,1,0,\dots,0]_x t^6-([2,0,0,1,0,\dots,0]_x+[0,1,0,1,0,\dots,0]_x \\
& +[1,0,0,0,1,0,\dots,0]_x)st^6-([1,1,1,0,0,\dots,0]_x+[0,0,2,0,0,\dots,0]_x \\
& +[2,0,0,1,0,\dots,0]_x+[0,1,0,1,0,\dots,0]_x+[1,0,0,0,1,0,\dots,0]_x)s^2t^6 \\
& +([1,1,0,0,0,\dots,0]_x+[0,0,0,1,1,0,\dots,0]_x)t^9-([3,0,1,0,0,\dots,0]_x \\
& +2[1,1,1,0,0,\dots,0]_x+[0,0,2,0,0,\dots,0]_x+2[2,0,0,1,0,\dots,0]_x \\
& +[0,1,0,1,0,0,\dots,0]_x+[1,0,0,0,1,0,\dots,0]_x)s^3t^6+\dots \\
& -([0,0,2,0,0,\dots,0]_x+[1,1,1,0,0,\dots,0]_x+[2,0,0,1,0,\dots,0]_x \\
& +[2,2,0,0,0,\dots,0]_x+[3,0,1,0,0,\dots,0]_x)s^4t^6+\dots \\
& -([1,1,1,0,\dots,0]_x+[2,2,0,0,\dots,0]_x+[3,0,1,0,\dots,0]_x)s^5t^6+\dots \\
& -[2,2,0,0,\dots,0]_x s^6t^6+\dots, \qquad (5.327)
\end{aligned}
$$

where we identify that the generators of the master space as,

$$
\begin{aligned}
s^2 &\rightarrow u_2 = \mathrm{Tr}(\phi^2)\\
s^3 &\rightarrow u_3 = \mathrm{Tr}(\phi^3)\\
[0,0,1,0,0,\dots,0]_x t^3 &\rightarrow B^{ijk} = \epsilon^{\alpha_1\alpha_2\alpha_3} Q^i_{\alpha_1} Q^j_{\alpha_2} Q^k_{\alpha_3}\\
[1,1,0,0,0,\dots,0]_x s t^3 &\rightarrow \begin{cases} {A_{001}}^{ijk} = \epsilon^{\alpha_1\alpha_2\alpha_3} Q^i_{\alpha_1} Q^j_{\alpha_2} \phi^{\beta}_{\alpha_3} Q^k_{\beta} \\ \epsilon_{ijkm_1\dots m_{N-3}} {A_{001}}^{ijk} = 0 \end{cases}\\
[1,1,0,0,0,\dots,0]_x s^2 t^3 &\rightarrow \begin{cases} {A_{002}}^{ijk} = \epsilon^{\alpha_1\alpha_2\alpha_3} Q^i_{\alpha_1} Q^j_{\alpha_2} \phi^{\beta_1}_{\alpha_3} \phi^{\beta_2}_{\beta_1} Q^k_{\beta_2} \\ \epsilon_{ijkm_1\dots m_{N-3}} {A_{002}}^{ijk} = -\frac{1}{3} u_2 \epsilon_{m_1\dots m_{N-3} rsu} B^{rsu} \\ {A_{011}}^{ijk} = \epsilon^{\alpha_1\alpha_2\alpha_3} Q^k_{\alpha_1} \phi^{\beta_1}_{\alpha_2} Q^i_{\beta_1} \phi^{\beta_2}_{\alpha_3} Q^j_{\beta_2} \\ \epsilon_{ijkm_1\dots m_{N-3}} {A_{011}}^{ijk} = \frac{1}{6} u_2 \epsilon_{m_1\dots m_{N-3} rsu} B^{rsu} \\ \rightarrow {A_{002}}^{ijk} = {A_{011}}^{ijk} + \frac{1}{2} u_2 B^{ijk} \end{cases}\\
[3,0,0,0,0,\dots,0]_x s^3 t^3 &\rightarrow \begin{cases} {A_{012}}^{ijk} = \epsilon^{\alpha_1\alpha_2\alpha_3} Q^i_{\alpha_1} \phi^{\beta_1}_{\alpha_2} Q^j_{\beta_1} \phi^{\beta_2}_{\alpha_3} \phi^{\beta_3}_{\beta_2} Q^k_{\beta_3} \\ {S_{012}}^{ijk} = {A_{012}}^{ijk} + {A_{012}}^{jki} + {A_{012}}^{kij} \end{cases} .
\end{aligned} \tag{5.328}
$$

Quadratic relations. The generalized quadratic relations amongst the generators of the 3 U(N) vortex master space are summarized in table 4 and table 5.

Vortex moduli space. Given the above summary of the quadratic relations between generators, we can express the vortex moduli space for 3 U(N) vortices as the following $\mathbb{C}^*$ projection,

$$
\begin{aligned}
\widetilde{\mathcal{V}}_{3,N} = \widetilde{\mathcal{F}^\flat}_{3,N} / \{ & B^{ijk} \simeq \lambda^3 B^{ijk}, {A_{001}}^{ijk} \simeq \lambda^3 {A_{001}}^{ijk},\\
& {A_{002}}^{ijk} \simeq \lambda^3 {A_{002}}^{ijk}, {S_{012}}^{ijk} \simeq \lambda^3 {S_{012}}^{ijk} \} .
\end{aligned} \tag{5.329}
$$

The master space of the vortex theory can be identified with the help of the quadratic relations as follows,

$$
\begin{aligned}
\widetilde{\mathcal{F}^\flat}_{3,N} = \mathbb{C}[u_2, u_3, B^{ijk}, {A_{001}}^{ijk}, {A_{002}}^{ijk}, {S_{012}}^{ijk}] / \{ &\\
& H_{i_1\dots i_{N-5}}^{\;k} = 0 ,\\
& {R_{(I)}}^{ij}_{\;\;k_1\dots k_{N-4}} = 0 , \dots , {R_{(III)}}^{i}_{\;\;j_1\dots j_{N-5}} = 0 ,\\
& {O_{(I)}}^{ij}_{\;\;k_1\dots k_{N-3}} = 0 , \dots , {O_{(VI)}}^{i}_{\;\;j_1\dots j_{N-5}} = 0 ,\\
& {P_{(I)}}^{ij}_{\;\;k_1\dots k_{N-4}} = 0 , \dots , {P_{(VII)}}^{ijk}_{\;\;m_1\dots m_{N-3}} = 0 ,\\
& {U_{(I)}}^{ij}_{\;\;k_1\dots k_{N-4}} = 0 , \dots , {U_{(VI)}}^{ijk}_{\;\;m_1\dots m_{N-3}} = 0 ,\\
& {V_{(I)}}^{ijk}_{\;\;m_1\dots m_{N-3}} = 0 , \dots , {V_{(III)}}^{ijk}_{\;\;m_1\dots m_{N-3}} = 0 ,\\
& Z^{ijklmn} = 0 \quad \} .
\end{aligned} \tag{5.330}
$$

The dimension of $\widetilde{\mathcal{V}}_{3,N}$ is $3N-1$.

Order t^6	Quadratic Relation
$-[1,0,0,0,1,0,\ldots,0]t^6$	$H_{i_1\ldots i_{N-5}}{}^k = \epsilon_{ijpqrsu}B^{pqr}B^{suk} = 0$
Order st^6	Quadratic Relation
$-[0,1,0,1,0,\ldots,0]_x st^6$	$R_{(I)}{}^{ij}{}_{k_1\ldots k_{N-4}} = A_{001}{}^{ijm}B^{pqr}\epsilon_{mpqrk_1\ldots k_{N-4}} = 0$
$-[2,0,0,1,0,\ldots,0]_x st^6$	$R_{(II)}{}^{ij}{}_{k_1\ldots k_{N-4}} = A_{001}{}^{imj}B^{pqr}\epsilon_{mpqrk_1\ldots k_{N-4}} = 0$
$-[1,0,0,0,1,0,\ldots,0]_x st^6$	$R_{(III)}{}^{i}{}_{j_1\ldots j_{N-5}} = A_{001}{}^{imn}B^{klp}\epsilon_{mnklpj_1\ldots j_{N-5}} = 0$
Order s^2t^6	Quadratic Relation
$-[0,1,0,1,0,\ldots,0]_x s^2t^6$	$O_{(I)}{}^{ij}{}_{k_1\ldots k_{N-3}} = A_{002}{}^{ijp}B^{qrs}\epsilon_{pqrsk_1\ldots k_{N-3}} = 0$
$-2[2,0,0,1,0,\ldots,0]_x s^2t^6$	$O_{(II)}{}^{ij}{}_{k_1\ldots k_{N-3}} = A_{001}{}^{mni}A_{001}{}^{pqj}\epsilon_{mnpqk_1\ldots k_{N-3}} = 0$ $O_{(III)}{}^{ij}{}_{k_1\ldots k_{N-3}} = A_{002}{}^{imj}B^{npq}\epsilon_{mnpqk_1\ldots k_{N-3}} = 0$
$-[0,0,2,0,0,\ldots,0]_x s^2t^6$	$O_{(IV)ijklmnu_1\ldots u_{N-6}v_1\ldots v_{N-6}} = o_{(I)ijklmnu_1\ldots u_{N-6}v_1\ldots v_{N-6}} - \frac{1}{9}o_{(II)ijklmnu_1\ldots u_{N-6}v_1\ldots v_{N-6}}$ $o_{(I)ijklmnu_1\ldots u_{N-6}v_1\ldots v_{N-6}} = \epsilon_{pqsijku_1\ldots u_{N-6}}\epsilon_{uvrlmnv_1\ldots v_{N-6}}A_{001}{}^{pqr}A_{001}^{uvs}$ $o_{(II)ijklmnu_1\ldots u_{N-6}v_1\ldots v_{N-6}} = u_2B^{pqs}B^{uvr}\epsilon_{pqsijku_1\ldots u_{N-6}}\epsilon_{uvrlmnv_1\ldots v_{N-6}}$
$-[1,1,1,0,0,\ldots,0]_x s^2t^6$	$O_{(V)}{}^{ijk}{}_{m_1\ldots m_{N-3}} = o_{(III)}{}^{ijk}{}_{m_1\ldots m_{N-3}} - \frac{1}{3}o_{(IV)}{}^{ijk}{}_{m_1\ldots m_{N-3}} = 0$ $o_{(III)}{}^{ijk}{}_{m_1\ldots m_{N-3}} = A_{001}{}^{pqk}A_{001}{}^{ijr}\epsilon_{pqrm_1\ldots m_{N-3}}$ $o_{(IV)}{}^{ijk}{}_{m_1\ldots m_{N-3}} = A_{002}{}^{ijk}B^{pqr}\epsilon_{pqrm_1\ldots m_{N-3}}$
$-[1,0,0,0,1,0,\ldots,0]_x s^2t^6$	$O_{(VI)}{}^{i}{}_{j_1\ldots j_{N-5}} = A_{002}{}^{pqi}B^{lmn}\epsilon_{pqlmnj_1-j_{N-5}} = 0$
Order s^3t^6	Quadratic Relation
$-[0,1,0,1,0,\ldots,0]_x s^3t^6$	$P_{(I)}{}^{ij}{}_{k_1\ldots k_{N-4}} = A_{001}{}^{ijm}A_{002}{}^{pqr}\epsilon_{mpqrk_1\ldots k_{N-4}} = 0$
$-2[2,0,0,1,0,\ldots,0]_x s^3t^6$	$P_{(II)}{}^{ij}{}_{k_1\ldots k_{N-4}} = A_{001}{}^{mni}A_{002}{}^{pqj}\epsilon_{mnpqk_1\ldots k_{N-4}} = 0$ $P_{(III)}{}^{ij}{}_{k_1\ldots k_{N-4}} = S_{012}{}^{ijm}B^{npq}\epsilon_{mnpqk_1\ldots k_{N-4}} = 0$
$-[0,0,2,0,0,\ldots,0]_x s^3t^6$	$P_{(IV)ijklmno_1\ldots o_{N-6}w_1\ldots w_{N-6}} = p_{(I)ijklmno_1\ldots o_{N-6}w_1\ldots w_{N-6}} - \frac{1}{9}p_{(II)ijklmno_1\ldots o_{N-6}w_1\ldots w_{N-6}}$ $p_{(I)ijklmno_1\ldots o_{N-6}w_1\ldots w_{N-6}} = A_{001}{}^{pqu}A_{002}{}^{rsv}\epsilon_{pqvijko_1\ldots o_{N-6}}\epsilon_{rsulmnw_1\ldots w_{N-6}}$ $p_{(II)ijklmno_1\ldots o_{N-6}w_1\ldots w_{N-6}} = u_3B^{pqu}B^{rsv}\epsilon_{pquijko_1\ldots o_{N-6}}\epsilon_{rsvlmnw_1\ldots w_{N-6}}$
$-2[1,1,1,0,0,\ldots,0]_x s^3t^6$	$P_{(V)}{}^{ijk}{}_{m_1\ldots m_{N-3}} = p_{(III)}{}^{ijk}{}_{m_1\ldots m_{N-3}} - p_{(IV)}{}^{ijk}{}_{m_1\ldots m_{N-3}} = 0$ $P_{(VI)}{}^{ijk}{}_{m_1\ldots m_{N-3}} = p_{(III)}{}^{ijk}{}_{m_1\ldots m_{N-3}} - \frac{1}{2}p_{(V)}{}^{ijk}{}_{m_1\ldots m_{N-3}} = 0$ $p_{(III)}{}^{ijk}{}_{m_1\ldots m_{N-3}} = A_{001}{}^{ijp}A_{002}{}^{qrk}\epsilon_{pqrm_1\ldots m_{N-3}}$ $p_{(IV)}{}^{ijk}{}_{m_1\ldots m_{N-3}} = A_{002}{}^{ijp}A_{001}{}^{qrk}\epsilon_{pqrm_1\ldots m_{N-3}}$ $p_{(V)}{}^{ijk}{}_{m_1\ldots m_{N-3}} = u_2A_{001}{}^{ijp}B^{qrk}\epsilon_{pqrm_1\ldots m_{N-3}}$
$-[3,0,1,0,0,\ldots,0]_x s^3t^6$	$P_{(VII)}{}^{ijk}{}_{m_1\ldots m_{N-3}} = p_{(VI)}{}^{ijk}{}_{m_1\ldots m_{N-3}} + \frac{1}{3}p_{(VII)}{}^{ijk}{}_{m_1\ldots m_{N-3}}$ $p_{(VI)}{}^{ijk}{}_{m_1\ldots m_{N-3}} = (A_{001}{}^{ipj}A_{002}{}^{qrk} + A_{001}{}^{jpk}A_{002}{}^{qri} + A_{001}{}^{kpi}A_{002}{}^{qrj})\epsilon_{pqrm_1\ldots m_{N-3}}$ $p_{(VII)}{}^{ijk}{}_{m_1\ldots m_{N-3}} = S_{012}{}^{ijk}B^{pqr}\epsilon_{pqrm_1\ldots m_{N-3}}$

Table 4. The quadratic relations for 3 U(N) vortices, for orders t^6 to s^3t^6 of the Hilbert series.

Order t^6	Quadratic Relation
$-[2,0,0,1,0,\dots,0]_x s^4 t^6$	$U_{(I)}{}^{ij}{}_{k_1\dots k_{N-4}} = A_{002}{}^{pqi} A_{002}{}^{mnj} \epsilon_{pqmnk_1\dots k_{N-4}} = 0$
$-[0,0,2,0,0,\dots,0]_x s^4 t^6$	$U_{(II)ijklmnu_1\dots u_{N-6}v_1\dots v_{N-6}} = u_{(I)ijklmnu_1\dots u_{N-6}v_1\dots v_{N-6}} - \frac{1}{3} u_{(II)ijklmnu_1\dots u_{N-6}v_1\dots v_{N-6}} = 0$ $u_{(I)ijklmnu_1\dots u_{N-6}v_1\dots v_{N-6}} = A_{002}{}^{pqr} A_{002}{}^{uvs} \epsilon_{pqsijku_1\dots u_{N-6}} \epsilon_{uvrlmnv_1\dots v_{N-6}}$ $u_{(II)ijklmnu_1\dots u_{N-6}v_1\dots v_{N-6}} = u_2 u_2 B^{pqr} B^{uvw} \epsilon_{pqrijku_1\dots u_{N-6}} \epsilon_{uvwlmnv_1\dots v_{N-6}}$
$-2[1,1,1,0,0,\dots,0]_x s^4 t^6$	$U_{(III)}{}^{ijk}{}_{m_1\dots m_{N-3}} = u_{(V)}{}^{ijk}{}_{m_1\dots m_{N-3}} - 2u_{(III)}{}^{ijk}{}_{m_1\dots m_{N-3}} + \frac{2}{9} u_{(VI)}{}^{ijk}{}_{m_1\dots m_{N-3}} = 0$ $U_{(IV)}{}^{ijk}{}_{m_1\dots m_{N-3}} = u_{(IV)}{}^{ijk}{}_{m_1\dots m_{N-3}} + \frac{1}{6} u_{(VI)}{}^{ijk}{}_{m_1\dots m_{N-3}} = 0$ $u_{(III)}{}^{ijk}{}_{m_1\dots m_{N-3}} = A_{002}{}^{ijp} A_{002}{}^{qrk} \epsilon_{pqrm_1\dots m_{N-3}}$ $u_{(IV)}{}^{ijk}{}_{m_1\dots m_{N-3}} = A_{001}{}^{ijp} S_{012}{}^{qrk} \epsilon_{pqrm_1\dots m_{N-3}}$ $u_{(V)}{}^{ijk}{}_{m_1\dots m_{N-3}} = u_2 A_{001}{}^{ijk} A_{001}{}^{pqr} \epsilon_{pqrm_1\dots m_{N-3}}$ $u_{(VI)}{}^{ijk}{}_{m_1\dots m_{N-3}} = u_3 A_{001}{}^{ijk} B^{pqr} \epsilon_{pqrm_1\dots m_{N-3}}$
$-[2,2,0,0,0,\dots,0]_x s^4 t^6$	$U_{(V)}{}^{ijklmn} = u_{(VIII)}{}^{ijklmn} + u_{(VIII)}{}^{ijmnkl} = 0$ $u_{(VII)}{}^{ijklmn} = A_{001}{}^{ijk} S_{012}{}^{lmn}$ $u_{(VIII)}{}^{ijklmn} = u_{(VII)}{}^{ijklmn} - u_{(VII)}{}^{ijlkmn} - u_{(VII)}{}^{ijklnm} + u_{(VII)}{}^{ijlknm}$
$-[3,0,1,0,0,\dots,0]_x s^4 t^6$	$U_{(VI)}{}^{ijk}{}_{m_1\dots m_{N-3}} = (A_{001}{}^{ipq} S_{012}{}^{rjk} + A_{001}{}^{jpq} S_{012}{}^{rki} + A_{001}{}^{jpq} S_{012}{}^{rij}) \epsilon_{pqrm_1\dots m_{N-3}} = 0$
Order $s^5 t^6$	Quadratic Relation
$-[1,1,1,0,0,\dots,0]_x s^5 t^6$	$V_{(I)}{}^{ijk}{}_{m_1\dots m_{N-3}} = v_{(I)}{}^{ijk}{}_{m_1\dots m_{N-3}} + \frac{1}{2} v_{(II)}{}^{ijk}{}_{m_1\dots m_{N-3}} = 0$ $v_{(I)}{}^{ijk}{}_{m_1\dots m_{N-3}} = A_{002}{}^{ijp} S_{012}{}^{qrk} \epsilon_{pqrm_1\dots m_{N-3}}$ $v_{(II)}{}^{ijk}{}_{m_1\dots m_{N-3}} = u_3 A_{001}{}^{ijp} A_{001}{}^{qrk} \epsilon_{pqrm_1\dots m_{N-3}}$
$-[2,2,0,0,0,\dots,0]_x s^5 t^6$	$V_{(II)}{}^{ijklmn} = v_{(V)}{}^{ijklmn} + v_{(VIII)}{}^{ijmnkl} = 0$ $v_{(III)}{}^{ijklmn} = (A_{002}{}^{ijk} - \frac{1}{2} u_2 B^{ijk}) S_{012}{}^{lmn}$ $v_{(IV)}{}^{ijklmn} = v_{(III)}{}^{ijklmn} - v_{(III)}{}^{ijlkmn} - v_{(III)}{}^{ijklnm} + v_{(III)}{}^{ijlknm}$ $v_{(V)}{}^{ijklmn} = v_{(IV)}{}^{ijklmn} - v_{(IV)}{}^{mnklij}$ $v_{(VI)}{}^{ijklmn} = u_3 A_{001}{}^{ijk} A_{001}{}^{lmn}$ $v_{(VII)}{}^{ijklmn} = v_{(VI)}{}^{ijklmn} - v_{(VI)}{}^{ijlkmn} - v_{(VI)}{}^{ijklnm} + v_{(VI)}{}^{ijlknm}$ $v_{(VIII)}{}^{ijklmn} = v_{(VII)}{}^{ijklmn} + v_{(VII)}{}^{klijmn}$
$-[3,0,1,0,0,\dots,0]_x s^5 t^6$	$V_{(III)}{}^{ijk}{}_{m_1\dots m_{N-3}} = v_{(X)}{}^{ijk}{}_{m_1\dots m_{N-3}} - \frac{1}{3} v_{(XII)}{}^{ijk}{}_{m_1\dots m_{N-3}} = 0$ $v_{(IX)}{}^{ijk}{}_{m_1\dots m_{N-3}} = A_{002}{}^{ipq} S_{012}{}^{rjk} \epsilon_{pqrm_1\dots m_{N-3}}$ $v_{(X)}{}^{ijk}{}_{m_1\dots m_{N-3}} = v_{(IX)}{}^{ijk}{}_{m_1\dots m_{N-3}} + v_{(IX)}{}^{jki}{}_{m_1\dots m_{N-3}} + v_{(IX)}{}^{kij}{}_{m_1\dots m_{N-3}}$ $v_{(XI)}{}^{ijk}{}_{m_1\dots m_{N-3}} = u_2 A_{001}{}^{ipj} A_{002}{}^{qrk} \epsilon_{pqrm_1\dots m_{N-3}}$ $v_{(XII)}{}^{ijk}{}_{m_1\dots m_{N-3}} = v_{(XI)}{}^{ijk}{}_{m_1\dots m_{N-3}} + v_{(XI)}{}^{jki}{}_{m_1\dots m_{N-3}} + v_{(XI)}{}^{kij}{}_{lmno}$
Order $s^6 t^6$	Quadratic Relation
$-[2,2,0,0,0,0]_x s^6 t^6$	$Z^{ijklmn} = z_{(II)}{}^{ijklmn} + z_{(II)}{}^{ijmnkl} = 0$ $z_{(I)}{}^{ijklmn} = S_{012}{}^{ijk} S_{012}{}^{lmn}$ $z_{(II)}{}^{ijklmn} = z_{(I)}{}^{ijklmn} - z_{(I)}{}^{ijlkmn} - z_{(I)}{}^{ijklnm} + z_{(I)}{}^{ijlknm}$

Table 5. The quadratic relations for 3 U(N) vortices, for orders $s^4 t^6$ to $s^6 t^6$ of the Hilbert series.

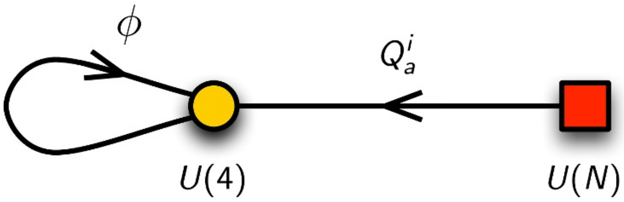

Figure 8. Quiver diagram of the 4 U(N) vortex theory.

6 4 U(N) vortices on $\mathbb{C}$

The quiver diagram of the 4 U(N) vortex theory is shown in figure 8.

The moduli space. The moduli space of 4 U(N) vortices $\mathcal{V}_{4,N}$ is a partial $\mathbb{C}^*$ quotient of the master space $\mathcal{F}^\flat_{4,N}$. The generators $x_1, \ldots, x_d$ of $\mathcal{F}^\flat_{4,N}$ can be considered as coordinates for the $\mathbb{C}^*$ projection which takes the form

$$(x_1, \ldots, x_d) \simeq (\lambda^{w_1} x_1, \ldots, \lambda^{w_d} x_d)\,. \tag{6.1}$$

Above, λ is the $\mathbb{C}^*$ parameter and $w_1, \ldots, w_d$ are respectively the U(1) weights for the coordinates $x_1, \ldots, x_d$. Accordingly, the vortex master space is $\mathbb{C}^*$ projected as follows,

$$\mathcal{V}_{4,N} = \mathcal{F}^\flat_{4,N} / \{x_1 \simeq \lambda^{w_1}, \ldots, x_d \simeq \lambda^{w_d}\}\,. \tag{6.2}$$

The Molien integral and Hilbert series. The Hilbert series of the 4 U(N) vortex master space is given by the following Molien integral,

$$g(t, s, x; \mathcal{F}^\flat_{4,N}) = \oint \mathrm{d}\mu_{\mathrm{SU}(4)}\ \mathrm{PE}\Big[[0,0,1]_w [1,0,\ldots,0]_x t + [1,0,1]_w s\Big]\,, \tag{6.3}$$

where Q^i_α transforms in $[0,0,1]_w[1,0,\ldots,0]_x t$ and ϕ transforms in $[1,0,1]_w s$. $\mathrm{d}\mu_{\mathrm{SU}(4)}$ is the SU(4) Haar measure.

Center of mass contribution. The integrand in (6.3) for the master space Hilbert series can be rewritten as follows,

$$\begin{aligned}
&\mathrm{PE}\Big[[0,0,1]_w[1,0,\ldots,0]_x t + [1,0,1]_w s\Big] = \\
&\quad \frac{1}{1-s}\mathrm{PE}\Big[[0,0,1]_w[1,0,\ldots,0]_x t + (w_1 w_3 + w_1 w_2 w_3^{-1} + w_1^{-1} w_2 w_3 + w_1^2 w_2^{-1} \\
&\qquad\qquad + w_2^{-1} w_3^2 + w_1^{-1} w_2^2 w_3^{-1} + 2 + w_1 w_2^{-2} w_3 + w_2 w_3^{-2} + w_1^{-2} w_2 + w_1 w_2^{-1} w_3^{-1} \\
&\qquad\qquad + w_1^{-1} w_2^{-1} w_3 + w_1^{-1} w_3^{-1}) s\Big]\,,
\end{aligned} \tag{6.4}$$

where the character of the adjoint of SU(4) is given by

$$\begin{aligned}
[1,0,1]_w \;=\;& w_1 w_3 + w_1 w_2 w_3^{-1} + w_1^{-1} w_2 w_3 + w_1^2 w_2^{-1} + w_2^{-1} w_3^2 + w_1^{-1} w_2^2 w_3^{-1} + 3 \\
& + w_1 w_2^{-2} w_3 + w_2 w_3^{-2} + w_1^{-2} w_2 + w_1 w_2^{-1} w_3^{-1} + w_1^{-1} w_2^{-1} w_3 + w_1^{-1} w_3^{-1}\,.
\end{aligned} \tag{6.5}$$

The $\frac{1}{1-s}$ prefactor in (6.5) does not interfere with the Molien integral and also is independent of the $\mathbb{C}^*$ projection of the vortex master space. It refers to the center of mass position of the 4 vortices.

6.1 4 U(1) vortices on $\mathbb{C}$

The Hilbert series of the 4 U(1) vortex master space is

$$g(t,s,x;\widetilde{\mathcal{F}^\flat}_{4,1}) = \frac{1}{(1-s^2)(1-s^3)(1-s^4)(1-s^6t^4)}\,. \tag{6.6}$$

The generalization of the master space Hilbert series for any k U(1) vortices is

$$g(t,s,x;\widetilde{\mathcal{F}^\flat}_{k,1}) = \frac{1}{(1-s^{k(k+1)/2}t^k)\prod_{i=2}^{k}(1-s^i)}\,. \tag{6.7}$$

The vortex master space for k U(1) vortices is

$$\widetilde{\mathcal{V}}_{k,1} = \mathbb{C}^{k-1}\,. \tag{6.8}$$

6.2 4 U(2) vortices on $\mathbb{C}$

The Hilbert series for the 4 SU(2) vortex moduli space is given by the following Molien integral,

$$g(t,s,x;\widetilde{\mathcal{F}^\flat}_{4,2}) = \oint \mathrm{d}\mu_{\mathrm{SU}(4)}\mathrm{PE}\Big[[0,0,1]_w[1]_x t + [1,0,1]_w s\Big]\,. \tag{6.9}$$

When solved, the above integral gives the Hilbert series

$$\begin{aligned} g(t,s,x;\widetilde{\mathcal{F}^\flat}_{4,2}) = &\frac{1}{(1-s^2)(1-s^3)(1-s^4)(1-s^2t^4)(1-s^3t^4)^2(1-s^6t^4)^2}\times \\ &(1+s^3t^4+4s^4t^4+3s^5t^4+3s^6t^4+3s^8t^8+3s^9t^8+4s^{10}t^8 \\ &+s^{11}t^8+s^{14}t^{12})\,, \end{aligned} \tag{6.10}$$

where we have set for simplicity the global SU(2) fugacity to $x = 1$. The base manifold is a non-complete intersection of dimension 9. As a character expansion the Hilbert series is

$$\begin{aligned} g(t,s,x;\widetilde{\mathcal{F}^\flat}_{4,2}) = &\frac{1}{(1-s^2)(1-s^3)(1-s^4)(1-s^2t^4)(1-s^4t^4)} \\ &\times\sum_{n_3=0}^{\infty}\sum_{n_6=0}^{\infty}\Big[[2n_3+4n_6]_x s^{3n_3+6n_6}t^{4n_3+4n_6} \\ &\qquad +[2n_3+4n_6+2]_x s^{3n_3+6n_6+4}t^{4n_3+4n_6+4} \\ &\qquad +[2n_3+4n_6+2]_x s^{3n_3+6n_6+5}t^{4n_3+4n_6+4} \\ &\qquad +[2n_3+4n_6+4]_x s^{3n_3+6n_6+9}t^{4n_3+4n_6+8}\Big]\,. \end{aligned} \tag{6.11}$$

The plethystic logarithm of the Hilbert series is

$$\begin{aligned} \mathrm{PL}\Big[g(t,s,x;\widetilde{\mathcal{F}^\flat}_{4,2})\Big] = &\; s^2+s^3+s^4+[2]_x s^3t^4+[2]_x s^4t^4+[2]_x s^5t^4+[4]_x s^6t^4 \\ &-s^6t^8-[2]_x s^8t^8-(2+[2]_x+[4]_x)s^8t^8-(1+2[2]_x+[4]_x)s^9t^8 \\ &-(1+[2]_x+2[4]_x)s^{10}t^8-([2]_x+[4]_x)s^{11}t^8 \\ &-(1+[4]_x)s^{12}t^8+\ldots\,. \end{aligned} \tag{6.12}$$

7 Summary of Hilbert series for vortices

7.1 Unrefined Hilbert series

We summarize in this section the Hilbert series of the vortex master spaces in unrefined form. This means, we set the fugacities corresponding to the global symmetry SU(N) to $x_i = 1$. The only remaining fugacities are t, corresponding to the remaining U(1) R-symmetry and the fugacity s corresponding to the U(1) residual gauge symmetry. The unrefined Hilbert series take the following form:

1 Vortex:

$$g(t,s;\widetilde{\mathcal{F}^\flat}_{1,1}) = \frac{1}{1-t} \tag{7.1}$$

$$g(t,s;\widetilde{\mathcal{F}^\flat}_{1,2}) = \frac{1}{(1-t)^2} \tag{7.2}$$

$$g(t,s;\widetilde{\mathcal{F}^\flat}_{1,3}) = \frac{1}{(1-t)^3}\,. \tag{7.3}$$

2 Vortices:

$$g(t,s;\widetilde{\mathcal{F}^\flat}_{2,1}) = \frac{1}{(1-s^2)(1-st^2)} \tag{7.4}$$

$$g(t,s;\widetilde{\mathcal{F}^\flat}_{2,2}) = \frac{1+st^2}{(1-s^2)(1-t^2)(1-st^2)^2} \tag{7.5}$$

$$g(t,s;\widetilde{\mathcal{F}^\flat}_{2,3}) = \frac{1+3st^2-3st^4-s^2t^6}{(1-s^2)(1-t^2)^3(1-st^2)^3} \tag{7.6}$$

$$\begin{aligned} g(t,s;\widetilde{\mathcal{F}^\flat}_{2,4}) = {} & \frac{1}{(1-s^2)(1-t^2)^5(1-st^2)^5} \times (1+t^2+5st^2-10st^4-5s^2t^4+st^6 \\ & -s^3t^6+5s^2t^8+10s^3t^8-5s^3t^{10}-s^4t^{10}-s^4t^{12}) \end{aligned} \tag{7.7}$$

$$\begin{aligned} g(t,s;\widetilde{\mathcal{F}^\flat}_{2,5}) = {} & \frac{1}{(1-s^2)(1-t^2)^7(1-st^2)^7} \times (1+3t^2+8st^2+t^4-21st^4-14s^2t^4 \\ & -7st^6-7s^2t^6+51s^2t^8+70s^3t^8+5s^4t^8-5s^2t^{10}-70s^3t^{10} \\ & -51s^4t^{10}+7s^4t^{12}+7s^5t^{12}+14s^4t^{14}+21s^5t^{14}-s^6t^{14} \\ & -8s^5t^{16}-3s^6t^{16}-s^6t^{18})\,. \end{aligned} \tag{7.8}$$

3 Vortices:

$$g(t,s;\widetilde{\mathcal{F}^\flat}_{3,1}) = \frac{1}{(1-s^2)(1-s^3)(1-s^3t^3)} \tag{7.9}$$

$$g(t,s;\widetilde{\mathcal{F}^\flat}_{3,2}) = \frac{1+2s^2t^3+2s^3t^3+s^5t^6}{(1-s^2)(1-s^3)(1-t^3)(1-st^3)^2(1-s^3t^3)^2} \tag{7.10}$$

$$\begin{aligned} g(t,s;\widetilde{\mathcal{F}^\flat}_{3,3}) = {} & \frac{1}{(1-s^2)(1-s^3)(1-t^3)(1-st^3)^4(1-s^3t^3)^4} \times (1+4st^3+8s^2t^3 \\ & +6s^3t^3+s^2t^6+5s^3t^6+6s^4t^6+3s^5t^6-6s^6t^6-s^4t^9-12s^5t^9 \\ & -15s^6t^9-15s^7t^9-12s^8t^9-s^9t^9-6s^7t^{12}+3s^8t^{12}+6s^9t^{12} \\ & +5s^{10}t^{12}+s^{11}t^{12}+6s^{10}t^{15}+8s^{11}t^{15}+4s^{12}t^{15}+s^{13}t^{18}) \end{aligned} \tag{7.11}$$

$$
\begin{aligned}
g(t,s;\widetilde{\mathcal{F}^\flat}_{3,4}) = \;& \frac{1}{(1-s^2)(1-s^3)(1-t^3)^4(1-st^3)^6(1-s^3t^3)^6} \times (1+14st^3+20s^2t^3 \\
& +14s^3t^3-16st^6+5s^2t^6+46s^3t^6+62s^4t^6+20s^5t^6-21s^6t^6 \\
& +4st^9-44s^2t^9-104s^3t^9-124s^4t^9-166s^5t^9-152s^6t^9 \\
& -190s^7t^9-100s^8t^9-4s^9t^9+21s^2t^{12}+25s^3t^{12}+15s^4t^{12} \\
& +153s^5t^{12}+135s^6t^{12}+101s^7t^{12}+110s^8t^{12}+85s^9t^{12} \\
& +139s^{10}t^{12}+61s^{11}t^{12}+10s^{12}t^{12}+14s^3t^{15}+34s^4t^{15}-20s^5t^{15} \\
& +156s^6t^{15}+396s^7t^{15}+264s^8t^{15}+248s^9t^{15}+256s^{10}t^{15} \\
& +292s^{11}t^{15}+116s^{12}t^{15}-26s^{13}t^{15}-2s^{14}t^{15}+s^4t^{18}+s^5t^{18} \\
& -109s^6t^{18}-275s^7t^{18}-218s^8t^{18}-460s^9t^{18}-788s^{10}t^{18} \\
& -788s^{11}t^{18}-460s^{12}t^{18}-218s^{13}t^{18}-275s^{14}t^{18}-109s^{15}t^{18} \\
& +s^{16}t^{18}+s^{17}t^{18}-2s^7t^{21}-26s^8t^{21}+116s^9t^{21}+292s^{10}t^{21} \\
& +256s^{11}t^{21}+248s^{12}t^{21}+264s^{13}t^{21}+396s^{14}t^{21}+156s^{15}t^{21} \\
& -20s^{16}t^{21}+34s^{17}t^{21}+14s^{18}t^{21}+10s^9t^{24}+61s^{10}t^{24} \\
& +139s^{11}t^{24}+85s^{12}t^{24}+110s^{13}t^{24}+101s^{14}t^{24}+135s^{15}t^{24} \\
& +153s^{16}t^{24}+15s^{17}t^{24}+25s^{18}t^{24}+21s^{19}t^{24}-4s^{12}t^{27}-100s^{13}t^{27} \\
& -190s^{14}t^{27}-152s^{15}t^{27}-166s^{16}t^{27}-124s^{17}t^{27}-104s^{18}t^{27} \\
& -44s^{19}t^{27}+4s^{20}t^{27}-21s^{15}t^{30}+20s^{16}t^{30}+62s^{17}t^{30}+46s^{18}t^{30} \\
& +5s^{19}t^{30}-16s^{20}t^{30}+14s^{18}t^{33}+20s^{19}t^{33}+14s^{20}t^{33}+s^{21}t^{36})\,. \qquad (7.12)
\end{aligned}
$$

7.2 Highest weight Hilbert series

We have in the sections above computed the Hilbert series for vortex master spaces in order to characterize their algebraic structure as well as to identify their generators for the $\mathbb{C}^*$ projection to the vortex moduli space. The Hilbert series were refined such that one had the following collection of fugacities,

$$
\begin{aligned}
s &\rightarrow \phi_{\alpha\beta} \\
t &\rightarrow Q^i_\alpha \\
[n_1,\ldots,n_{N-1}]_x &\rightarrow \mathrm{SU}(N) \;\; \text{global symmetry}\,. \qquad (7.13)
\end{aligned}
$$

Let us summarize the character expansions for the master space Hilbert series for $1,2,3$ $\mathrm{U}(N)$ vortices,

$$
\begin{aligned}
g(t,s,x;\widetilde{\mathcal{F}^\flat}_{1,N}) &= \sum_{n_0=0}^{\infty}[n_0,0,\ldots,0]_x t^{n_0}\,, \\
g(t,s,x;\widetilde{\mathcal{F}^\flat}_{2,N}) &= \sum_{n_0=0}^{\infty}\sum_{n_1=0}^{\infty}[2n_1,n_0,0,\ldots,0]_x s^{n_1}t^{2(n_0+n_1)}\,,
\end{aligned}
$$

$$g(t,s,x;\widetilde{\mathcal{F}^\flat}_{3,N}) = \frac{1}{(1-s^2)(1-s^3)} \times$$
$$\sum_{n_0=0}^{\infty}\sum_{n_1=0}^{\infty}\sum_{n_2=0}^{\infty}\sum_{n_3=0}^{\infty}\Big[[n_1+n_2+3n_3,n_1+n_2,n_0,0,\dots,0]s^{n_1+2n_2+3n_3}t^{3n_0+3n_1+3n_2+3n_3}$$
$$+[n_1+n_2,n_1+n_2+3n_3+3,n_0,0,\dots,0]s^{n_1+2n_2+3n_3+3}t^{3n_0+3n_1+3n_2+6n_3+6}\Big]. \quad (7.14)$$

Highest weight Hilbert series. We now use a more compact form of writing characters of irreducible representations in a Hilbert series [34]. Given that characters of the form $[n_1,\dots,n_r]$ are written in terms of highest weight Dynkin labels, we introduce for each of the r labels its own fugacity μ_i such that

$$[n_1,n_2,\dots,n_r]_G \to \prod_{i=1}^{r}\mu_i^{n_i} = \mu_1^{n_1}\mu_2^{n_2}\dots\mu_r^{n_r}\,, \quad (7.15)$$

for a group G of rank r. Effectively, the above map replaces a character with a product of fugacities carrying as exponents the highest weight Dynkin labels of the irreducible representation of the group G.

There are various motivations for introducing the above map. One of them is the ability to write character expansions of Hilbert series given by infinite concatenated sums as compact rational functions. For the character expansion in (7.14) corresponding to the Hilbert series of $1,2,3$ $\mathrm{U}(N)$ vortex master spaces, the highest weight forms are as follows

$$\begin{aligned}
g(t,s,x;\widetilde{\mathcal{F}^\flat}_{1,N}) &\to \frac{1}{1-\mu_1 t}\,,\\
g(t,s,x;\widetilde{\mathcal{F}^\flat}_{2,N}) &\to \frac{1}{(1-s^2)(1-\mu_2 t^2)(1-\mu_1^2 s t^2)}\,,\\
g(t,s,x;\widetilde{\mathcal{F}^\flat}_{3,N}) &\to \frac{1+\mu_1\mu_2 s^2t^3+\mu_1^2\mu_2^2 s^4t^6}{(1-s^2)(1-s^3)(1-\mu_3 t^3)(1-\mu_1\mu_2 s t^3)(1-\mu_1^3 s^3t^3)(1-\mu_2^3 s^3 t^6)}\,.
\end{aligned} \quad (7.16)$$

8 Conclusions

With this work, we have classified fully for the first time the moduli spaces for supersymmetric gauge theories of up to 3 $\mathrm{U}(N)$ vortices. This was done by describing the vortex moduli space as a partially weighted projective space coming from a $\mathbb{C}^*$ projection of the vortex master space. In our classification, we have given the full algebraic structure of the vortex master spaces by identifying all of their generators and quadratic relations formed amongst the generators. The information given by the residual $\mathrm{U}(1)$ gauge charges carried by the generators allows us to describe the projection into the full vortex moduli space.

Our results for 2 vortices agree with previous results in [3, 7]. In view of our complete analysis of moduli spaces for 3 vortices and the preliminary computations we have presented here for 4 vortices, it would be interesting to generalize our results to higher number of vortices. With our current methods and tools, this task seems to be a challenge at this moment.

We have seen with the computation of the Hilbert series for vortex master spaces that its expression as an infinite expansion in terms of characters of $\mathrm{SU}(N)$ is an increasingly difficult problem to solve. The most evident example is the Hilbert series of the master space for 4 U(2) vortices which for the purpose of our argument we present again as follows,

$$
\begin{aligned}
g(t,s,x;\widetilde{\mathcal{F}^\flat}_{4,2}) = {} & \frac{1}{(1-s^2)(1-s^3)(1-s^4)(1-s^2t^4)(1-s^4t^4)} \\
& \times \sum_{n_3=0}^{\infty}\sum_{n_6=0}^{\infty}\Big[[2n_3+4n_6]_x s^{3n_3+6n_6}t^{4n_3+4n_6} \\
& \qquad +[2n_3+4n_6+2]_x s^{3n_3+6n_6+4}t^{4n_3+4n_6+4} \\
& \qquad +[2n_3+4n_6+2]_x s^{3n_3+6n_6+5}t^{4n_3+4n_6+4} \\
& \qquad +[2n_3+4n_6+4]_x s^{3n_3+6n_6+9}t^{4n_3+4n_6+8}\Big]\,.
\end{aligned} \tag{8.1}
$$

As discussed in section 7.2, we use highest weight fugacities. For the Hilbert series of the master space of 4 U(2) vortices above, the new fugacity is μ for SU(2) such that the representations are mapped to

$$
[n]_x \to \mu^n\,. \tag{8.2}
$$

The above map dramatically simplifies the expression in (8.1) to

$$
\begin{aligned}
g(t,s,x;\widetilde{\mathcal{F}^\flat}_{4,2}) \to {} & \frac{1}{(1-s^2)(1-s^3)(1-s^4)(1-s^2t^4)(1-s^4t^4)} \\
& \times \frac{1+\mu^2s^4t^4+\mu^2s^5t^4+\mu^4s^9t^8}{(1-\mu^2s^3t^4)(1-\mu^4s^6t^4)}\,.
\end{aligned} \tag{8.3}
$$

The expression becomes a rational function, which of great interest has a palindrome as its numerator. This is not the actual Hilbert series itself. This can be seen by comparing the so called highest weight Hilbert series in (8.3) with the actual unrefined Hilbert series

$$
\begin{aligned}
g(t,s;\widetilde{\mathcal{F}^\flat}_{4,2}) = {} & \frac{1}{(1-s^2)(1-s^3)(1-s^4)(1-s^2t^4)(1-s^4t^4)(1-s^3t^4)^2(1-s^6t^4)^2} \\
& \times(1+s^3t^4+3s^4t^4+3s^5t^4+3s^6t^4-s^7t^8-s^8t^8+s^{10}t^8+s^{11}t^8 \\
& -3s^{12}t^{12}-3s^{13}t^{12}-3s^{14}t^{12}-s^{15}t^{12}-s^{18}t^{16})\,.
\end{aligned} \tag{8.4}
$$

From here, it can be seen that the map in (8.2) transforms a palindromic Hilbert series of a non-compact Calabi-Yau space to another different rational function with a palindromic numerator. It is of great interest to explore the meaning of the function in (8.3) in comparison to the original vortex Hilbert series in (8.4).

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